

Homogeneous $\mathbb{Z}/2$ -harmonic forms and spinors on $\mathbb{R}^4$ from regular 4-polytopes

Clifford Taubes and Yingying Wu

ABSTRACT. We describe novel local singularity models for $\mathbb{Z}/2$ harmonic 1-forms, self-dual 2-forms and spinors in dimension 4. These models are homogeneous versions on $\mathbb{R}^4$ whose singular sets are cones on the 1-skeleta of certain regular 4-dimensional polytopes.

CONTENTS

Part 1. Analytic theory of $\mathbb{Z}/2$ -harmonic forms on $S^3 \setminus \Gamma$	356
1. Introduction	356
2. Homogeneous, harmonic on $\mathbb{R}^4$ from eigensections on S^3	357
3. Some topology regarding the set Z and the line bundle $\mathcal{I}$	361
4. Some initial analysis	362
5. Equivariant analysis	367
6. Hölder continuity across the interior of the edges of Γ	371
7. Almgren's frequency function	383
8. Proofs for Lemmas 4.2 and 4.4	393
9. Fourier modes of ψ near an edge of Γ	399
10. Proof of Theorem 1.2	403
Part 2. Polytope symmetry, equivariance, and topological invariants	417
11. Regular 4-polytopes and their symmetries	417
12. Monodromy, connected double covers, and fiberwise lifts	424
13. Order-3 isometries on regular 5-cell	430
14. Order-3 isometries on regular 8-cell	435
15. Order-3 isometries on regular 24-cell	441
16. Order-3 isometries on regular 120-cell	445
17. Order-5 isometries on regular 600-cell	449
References	454

Part 1. Analytic theory of $\mathbb{Z}/2$ -harmonic forms on $S^3 \setminus \Gamma$

1. Introduction

Homogeneous, $\mathbb{Z}/2$ harmonic differential forms and spinors on $\mathbb{R}^4$ arise as rescaling limits of divergent sequences to certain 4-dimensional generalizations of the Seiberg–Witten and (anti)-self-dual Yang–Mills equations. (See e.g. [22, 24, 25].) What follows momentarily is a working definition of these objects of interest. First, some preliminaries to set the stage: The set of length 1 vectors in $\mathbb{R}^4$ is the 3-sphere, denoted here by S^3 . The radial projection map from $\mathbb{R}^4 \setminus \{0\}$ to S^3 that sends any given non-zero vector x to the unit vector along the ray from the origin to x is denoted by π . The multiplicative group $(0, \infty)$ acts on $\mathbb{R}^4$ as the 1-parameter family of coordinate rescaling diffeomorphisms using the rule whereby any given $\lambda \in (0, \infty)$ sends any given vector x to λx . Next, some notation: The upcoming definition refers to a “Euclidean line bundle”. This is a real line bundle with fiber metric. Given that metric, there is a canonical definition of a (covariant) derivative for sections of the line bundle (and thus for differential form/spinor valued sections). This is because the line bundle is locally isometric to the product $\mathbb{R}$ -bundle with the isometry being ambiguous only up to multiplication by ± 1 , which commutes with differentiation.

DEFINITION 1.1. A homogeneous, $\mathbb{Z}/2$ harmonic differential form or spinor on $\mathbb{R}^4$ is characterized by a data set $(Z, \mathcal{I}, v)$ consisting of the following elements:

- What is denoted by Z signifies the inverse image via π of a closed, non-empty set in S^3 with finite 1-dimensional Hausdorff measure in S^3 .
- What is denoted by $\mathcal{I} \rightarrow \mathbb{R}^4 - Z$ signifies the pull-back via π of a real (Euclidean) line bundle defined on the complement of $Z \cap S^3$ which has no extension to any ball in S^3 that contains any point of $Z \cap S^3$.
- What is denoted by v signifies an $\mathcal{I}$ -valued harmonic, differential form or spinor defined over $\mathbb{R}^4 - Z$ that pulls-back as a non-zero multiple of itself via the action of $(0, \infty)$ on $\mathbb{R}^4$. In addition, the conditions below must be met:
 - a) The norm of v extends over Z to define a Hölder continuous function on $\mathbb{R}^4$ whose zero locus contains Z .
 - b) The square of the norm of the first derivative of any components of v has finite integral over the complement of Z in any ball in $\mathbb{R}^4$.

Of particular interest in this paper are the cases where v is a 1-form, self-dual 2-form, or self-dual spinor. (The self-dual spinor bundle on $\mathbb{R}^4$ is the product, rank 2, complex vector bundle $\mathbb{R}^4 \times \mathbb{C}^2$.)

To put the upcoming theorems in a historical context, note first that examples of homogeneous $\mathbb{Z}/2$ harmonic differential forms and spinors on

$\mathbb{R}^3$ were constructed previously, by the authors (see [26] and also [27]), by [3] and by [9]. As noted in [23], these objects on $\mathbb{R}^3$ can be pulled up to $\mathbb{R}^4$ via the projection map to any given hyperplane and then those pull-backs give examples on $\mathbb{R}^4$. In addition, there are examples whereby Z is the zero locus of a homogeneous, holomorphic function on $\mathbb{C}^2$ (given some isometry between $\mathbb{C}^2$ and $\mathbb{R}^4$). See for example [23] and [15]. In the former case, Z is the union of half-planes in $\mathbb{R}^4$ through the origin that share a common axis; and in the latter case, Z is a union of complex lines through the origin.

With the preceding as background, what follows is our main theorem.

THEOREM 1.2. *There exist homogeneous, $\mathbb{Z}/2$ harmonic 1-forms, self-dual 2-forms and self-dual spinors on $\mathbb{R}^4$ with the set Z being the cone on the union of the edges and vertices of the following 4-dimensional analogs of the Platonic solids: The 4-dimensional tetrahedron (the 5-cell), the 4-dimensional cube (the 8-cell), and the polytopes known as the 24-cell, the 120-cell and the 600-cell.*

See Part II for a detailed description of these polytopes.

In the cases of our theorem, the set Z is the union of the cones on geodesic arcs as is the case for the pull-backs from $\mathbb{R}^3$ and for the holomorphic examples. However, in no case is Z a union of half-planes and/or full planes. (The upcoming Section 11-17 gives a detailed description of the polytopes that are mentioned in the theorem.) By way of a look ahead to the proof: Our constructions exploit the symmetry groups of these 4-dimensional polytopes in the manner of the constructions in [26] which exploit the symmetry groups of the Platonic solids. Even so, the group theory here is different (and pretty) and the analysis has some novelties. (The constructions for $\mathbb{R}^3$ in [3] and [9] also exploit symmetry groups, but in a more sophisticated way.) An almost contemporaneous preprint [8] develops related microlocal methods in the study of two-valued harmonic functions and branched minimal submanifolds with stratified branching sets relevant to some of the local regularity and asymptotic questions considered here. Other recent papers with related results on $\mathbb{Z}/2$ harmonic 1-forms and spinors are the paper by Bera and Walpuski [1] and the paper by Jiahuang Chen [2].

2. Homogeneous, harmonic on $\mathbb{R}^4$ from eigensections on S^3

The basic observation (which is truly ancient) is that homogeneous, harmonic differential forms and spinors on $\mathbb{R}^4$ come from eigensections of corresponding differential operators acting on sections of some vector bundle over S^3 . To elaborate: The assertion that ν is homogeneous says in effect that there exists $\kappa \in \mathbb{R}$ such that the pull-back of ν under a rescaling diffeomorphism $x \mapsto \lambda x$ is equal to $\lambda^\kappa \nu$ for any given $\lambda \in (0, \infty)$. Note in this regard that Z is invariant under these coordinate rescalings, so these diffeomorphisms map $\mathbb{R}^4 - Z$ to itself. Meanwhile, the fact that $\mathcal{I}$ is pulled up from $S^3 - (Z \cap S^3)$ implies that $\mathcal{I}$ at any given point in $\mathbb{R}^4 - Z$ is canonically isomorphic to the pull-back of $\mathcal{I}$ via any rescaling diffeomorphism. This is

why it makes sense to say that the λ -pull-back of ν at a given point in $\mathbb{R}^4 - Z$ is equal to $\lambda^\kappa \nu$ at that point. In particular, homogeneity of ν implies that ν is completely determined by its restriction to $S^3 - (Z \cap S^3)$. This homogeneity also implies the following: Let r denote the radial coordinate $x \mapsto |x|$. The Lie derivative of ν along the vector field $r\partial_r$ (which generates the 1-parameter family of rescaling diffeomorphisms) is equal to $\kappa\nu$. (In the case when ν is a spinor, this Lie derivative is defined using the product structure on the appropriate vector bundle that is defined by parallel transport along rays from the origin using the Euclidean metric's Levi-Civita connection.) As explained momentarily, this property of the Lie derivative has fundamental implications when ν is a harmonic differential form or spinor.

A simple case to consider first is the case when ν is an $\mathcal{I}$ -valued, homogeneous, harmonic function. The assertion that ν is harmonic (thus $\Delta\nu = 0$ with Δ denoting the sum of the second derivatives in the four coordinate directions) is equivalent to the assertion that ν 's restriction to $S^3 - (Z \cap S^3)$ is an $\mathcal{I}$ -valued eigensection of the standard Laplacian on $S^3 - (Z \cap S^3)$ with eigenvalue $\kappa(\kappa + 2)$, thus

$$(2.1) \quad -\Delta^\perp \nu = \kappa(\kappa + 2)\nu$$

where $\Delta^\perp$ denotes here and subsequently the standard Laplacian on the 3-sphere. (The Laplace eigensections for the $S^2 - (Z \cap S^2)$ analog are studied in [26, 27].)

As explained in what follows, any given $\mathcal{I}$ -valued, homogeneous harmonic function on $\mathbb{R}^4 - Z$ with the norm of its exterior derivative extending across Z as a Hölder continuous function that vanishes on Z can be used to construct homogeneous, $\mathbb{Z}/2$ harmonic 1-forms and spinors for the same Z (but not directly any homogeneous, $\mathbb{Z}/2$ harmonic, self-dual 2-forms). To elaborate, let ϕ denote such a function. Then $d\phi$ is a homogeneous, $\mathbb{Z}/2$ harmonic 1-form for the given set Z . With regards to self-dual spinors: Let η denote a constant anti-self-dual spinor and let $\mathfrak{D}$ denote the standard Dirac operator on $\mathbb{R}^4$. Then $\mathfrak{D}(\phi\eta)$ is a homogeneous, $\mathbb{Z}/2$ harmonic, self-dual spinor for that same locus Z .

These last two examples are instances of a general construction: Let $\mathbb{V}_0$ denote a vector space (real or complex); and let $\{\gamma_1, \dots, \gamma_4\}$ denote a set of anti-symmetric (or skew-adjoint if $\mathbb{V}_0$ is complex) endomorphisms of $\mathbb{V}_0$ that obey the Clifford algebra relations

$$\begin{aligned} &\bullet \gamma_i \gamma_j + \gamma_j \gamma_i = 0 \text{ for each distinct index pair,} \\ &\bullet \gamma_i^2 = -\mathbb{I} \text{ for each } i. \end{aligned} \quad (2.2)$$

(Here and below, $\mathbb{I}$ denotes the identity endomorphism of $\mathbb{V}_0$.) Let $\{\partial_i\}_{i=1,2,3,4}$ denote the directional derivatives along the coordinate axes in $\mathbb{R}^4$ acting component-wise on sections of the product vector bundle $\mathbb{R}^4 \times \mathbb{V}_0$. Set

$$(2.3) \quad \mathfrak{D} = \gamma_1 \partial_1 + \dots + \gamma_4 \partial_4.$$

This is a first-order differential operator acting on sections of $\mathbb{R}^4 \times \mathbb{V}_0$ obeying $\mathfrak{D}^2 = -\Delta \mathbb{I}$ with Δ denoting the standard Laplacian acting component-wise also. Take ϕ as in the previous paragraph and let η denote any given element in $\mathbb{V}_0$. Then

$$(2.4) \quad v = \mathfrak{D}(\phi\eta)$$

is annihilated by $\mathfrak{D}$; and as such, it defines a $\mathbb{Z}/2$ homogeneous, harmonic (with respect to $\mathfrak{D}$) section of $\mathbb{V}$ (it is a section of the vector bundle $(\mathbb{R}^4 \times \mathbb{V}_0) \otimes \mathcal{I}$ on $\mathbb{R}^4 - Z$ that obeys the conditions set forth by the third bullet of Definition 1.1).

By way of a parenthetical remark: Any $\mathfrak{D}$ -harmonic section of $(\mathbb{R}^4 \times \mathbb{V}_0) \otimes \mathcal{I}$ can be written as a linear combination of sections of the form depicted in (2.4) if one is not concerned with the behavior along Z . To elaborate, let $(x_1, \dots, x_4)$ denote the Euclidean coordinates on $\mathbb{R}^4$, and let

$$(2.5) \quad x \cdot \gamma = x_1\gamma_1 + \dots + x_4\gamma_4.$$

The crucial observation is that if $\mathfrak{D}\psi = 0$ and ψ is homogeneous, thus $r\partial_r\psi = \kappa\psi$ with $\kappa \geq 0$, then

$$(2.6) \quad \psi = -(4 + 2\kappa)^{-1} \mathfrak{D}(x \cdot \gamma \psi).$$

Even so, it is not clear *a priori* that the norms of the derivatives of the components of $(x \cdot \gamma) \psi$ must extend across Z as Hölder continuous functions that vanish on Z if the norm of ψ does.

The construction of homogeneous, $\mathbb{Z}/2$ harmonic self-dual 2-forms requires a different approach. (Yet, examples of homogeneous, harmonic, $\mathcal{I}$ -valued solutions to the equation $*dw + du = 0$ for a pair $\nu = (w, u)$ of $\mathcal{I}$ -valued self-dual 2-form and $\mathcal{I}$ -valued function can be obtained via an appropriate version of (2.4). However, because the function component of ν won't be identically zero, the self-dual 2-form part isn't harmonic.)

To say more about the upcoming construction for $\mathbb{Z}/2$ harmonic, self-dual 2-forms, let ν denote for the moment a harmonic, $\mathcal{I}$ -valued self-dual 2-form on $\mathbb{R}^4 - Z$ for some scaling-invariant, codimension-2 set Z . Let $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ denote an oriented, orthonormal basis for T^*S^3 . The self-dual 2-form ν can be written schematically using this basis as

$$(2.7) \quad \nu = \nu_a \left(dr \wedge r \pi^* \hat{e}_a + \frac{1}{2} r^2 \varepsilon^{abc} \pi^* (\hat{e}^b \wedge \hat{e}^c) \right)$$

with the notation as follows: The repeated indices are implicitly summed over $\{1, 2, 3\}$ with it understood that $\{\nu_1, \nu_2, \nu_3\}$ denote $\mathcal{I}$ -valued functions on $\mathbb{R}^4 - Z$. Meanwhile, ε^{abc} denotes the totally anti-symmetric 3-tensor with $\varepsilon^{123} = 1$.

The self-dual 2-form ν is homogeneous and harmonic if and only if the three conditions below are met by $\{\nu_1, \nu_2, \nu_3\}$. This list uses $\nu^\perp$ to denote the 1-form

$$(2.8) \quad \nu^\perp = \nu_a \pi^* \hat{e}^a.$$

The symbol $*$ below denotes the metric Hodge star on S^3 :

$$\begin{aligned} \bullet \quad r\partial_r\nu^\perp &= (\kappa - 2)\nu^\perp, \\ \bullet \quad *d^\perp\nu^\perp &= \kappa\nu^\perp, \\ \bullet \quad d^\perp * \nu^\perp &= 0. \end{aligned} \tag{2.9}$$

These bullets say in effect that $\nu^\perp$ is the product of $r^{\kappa-2}$ times the pull-back from $S^3 - (S^3 \cap Z)$ via the radial projection π of an eigensection of the operator $*d$ acting on coclosed, $\mathcal{I}$ -valued 1-forms on $S^3 - (S^3 \cap Z)$.

To put (2.9) in some sort of perspective and to motivate some of the focus in the subsequent sections of this paper, reintroduce the vector space $\mathbb{V}_0$ and then the operator

$$(2.10) \quad \mathfrak{D} = \gamma_1\partial_1 + \gamma_2\partial_2 + \gamma_3\partial_3 + \gamma_4\partial_4$$

which acts on sections over $\mathbb{R}^4 - Z$ of the tensor product of the line bundle $\mathcal{I}$ with the product $\mathbb{V}_0$ -bundle. Let

$$(2.11) \quad \gamma_r = r^{-1}(x_1\gamma_1 + \cdots + x_4\gamma_4)$$

which is an endomorphism of that bundle with square equal to -1 times the identity endomorphism. The equation $\mathfrak{D}\nu = 0$ can be written alternately in the schematic form

$$(2.12) \quad r\partial_r\nu = \mathcal{D}\nu$$

with $\mathcal{D}$ being a homogeneous, first-order, elliptic differential operator that differentiates only in directions tangential to the constant r -spheres in $\mathbb{R}^4$. The assumption that ν is homogeneous implies that $r\partial_r\nu$ is equal to $\lambda\nu$ with λ constant. Hence, ν is both harmonic and homogeneous if and only if its restriction to the standard $r = 1$ sphere is an eigensection of the operator $\mathcal{D}$ over $S^3 - (S^3 \cap Z)$. Conversely, if ν is an eigensection of $\mathcal{D}$ on $S^3 - (S^3 \cap Z)$ with eigenvalue λ , and if the norm of ν vanishes on Z and is Hölder continuous across $S^3 \cap Z$, then $r^\lambda\pi^*\nu$ will be a homogeneous, harmonic (with respect to $\mathfrak{D}$) section of $(\mathbb{R}^4 \times \mathbb{V}_0) \otimes \mathcal{I}$ over $\mathbb{R}^4 - Z$ that is described by the third bullet of Definition 1.1.

By way of an example: Take $\mathfrak{D}$ to be the Dirac operator acting on $\mathbb{R}^4$'s spinor bundle, this being the $\mathbb{V}_0 = \mathbb{C}^2 \oplus \mathbb{C}^2$ case (the left-hand $\mathbb{C}^2$ summand gives the self-dual spinor bundle and the right-hand summand gives the anti-self-dual spinor bundle). A self-dual, $\mathcal{I}$ -valued spinor is homogeneous and harmonic if

$$(2.13) \quad \nu = r^\lambda\pi^*\psi$$

with ψ an eigensection of the 3-sphere's Dirac operator on $\mathcal{I}$ -valued spinors over $S^3 - (S^3 \cap Z)$ with eigenvalue $\lambda + \frac{3}{2}$. (The relevant version of $\mathfrak{D}$ differs from that Dirac operator by a constant multiple of the identity endomorphism.)

3. Some topology regarding the set Z and the line bundle $\mathcal{I}$

For the purposes of what is to come, the set Z will be the cone in $\mathbb{R}^4$ on certain sorts of embedded graphs in S^3 . To elaborate: Let $\Gamma \subset S^3$ denote a non-empty, connected graph with each vertex having even valency (which is to say that an even number of edges end at each vertex). The implicit assumption is that the edges of Γ are closed subsets of embedded arcs in S^3 , and the tangent vectors to the edges that are incident to any given vertex of Γ are distinct. (There will be an extra assumption later to the effect that the edges of Γ are geodesic arcs.) The upcoming Lemma 3.1 makes a formal statement to the effect that an appropriate line bundle $\mathcal{I}$ on $S^3 - \Gamma$ exists if and only if each vertex of Γ has even valency. The lemma refers to a *linking circle* of Γ . This is an embedded circle in $S^3 - \Gamma$ bounding a disk in S^3 that intersects Γ in a single point, that being a transversal intersection with the interior of an edge of Γ .

LEMMA 3.1. Let Γ denote a graph in S^3 . There is a class in $H^1(S^3 - \Gamma; \mathbb{Z}/2)$ with non-zero restriction to any linking circle of Γ if and only if each vertex of Γ has even valency. This is to say that there is a real line bundle over $S^3 - \Gamma$, which restricts as the Möbius line bundle to each linking circle of Γ .

This lemma is proved momentarily. The line bundle $\mathcal{I}$ for the constructions that follow will always be the line bundle from Lemma 3.1. By way of relevant instances for Lemma 3.1: The image in S^3 via the radial projection map π of the union of the edges and vertices of any of the polytopes from Theorem 1.2 is a graph in S^3 whose vertices all have even valency, this being four for the 5-cell, 8-cell and 120-cell, and eight and twelve in the respective cases of the 24-cell and 600-cell.

Proof of Lemma 3.1. Since real line bundles are classified by their first Stiefel–Whitney class, and since any $\mathbb{Z}/2\mathbb{Z}$ cohomology class is the first Stiefel–Whitney class of a real line bundle, it is sufficient to prove the lemma’s assertion regarding $H^1(S^3 - \Gamma; \mathbb{Z}/2)$. To this end, let g denote the rank of $H_1(S^3 - \Gamma; \mathbb{Z})$. A regular neighborhood of Γ is a handlebody whose boundary is homeomorphic to a Riemann surface with genus equal to g . (This surface can be built in the following way: Let V denote the set of vertices of Γ and let E denote the set of edges. Use each vertex to label a 2-sphere and each edge to label a cylinder. The ends of the cylinders are then attached to the complements of disjoint disks in the 2-spheres using the edge–vertex attachment rule that defines Γ .) In particular, since the regular neighborhood is a handlebody which deformation retracts to Γ , it follows that the first homology of $S^3 - \Gamma$ with $\mathbb{Z}/2$ coefficients has the same rank as the first homology of Γ ; thus a direct sum of g copies of $\mathbb{Z}/2$.

To continue, let Σ denote the boundary of a regular neighborhood of Γ in $S^3 - \Gamma$. This surface separates S^3 into two parts, one that contains Γ and the other that is disjoint from Γ ; it is a deformation retract of $S^3 - \Gamma$. The

“interior of Σ ” will denote the part that contains Γ . Let $H_\Gamma \subset H_1(\Sigma; \mathbb{Z})$ denote the kernel of the inclusion map to $H_1(S^3 - \Gamma; \mathbb{Z})$. The inclusion of Σ into the closure of its interior identifies H_Γ with $H_1(\Gamma; \mathbb{Z})$. Let $H_\Gamma^\perp$ denote the symplectic dual of H_Γ in $H_1(\Sigma; \mathbb{Z})$. This is the kernel of the inclusion map sending $H_1(\Sigma; \mathbb{Z})$ to the first homology of the interior of Σ . (The intersection form on $H_1(\Sigma; \mathbb{Z})$ defines a non-degenerate, skew-symmetric form on $H_1(\Sigma; \mathbb{Z})$.) The inclusion map identifies $H_\Gamma^\perp$ with $H_1(S^3 - \Gamma; \mathbb{Z})$.

With the preceding understood: The assertion of the lemma follows if there is a surface in $S^3 - \Gamma$ that intersects Σ transversally and, in this regard, has intersection number 1 with each generator of $H_\Gamma^\perp$. The task is to find this surface when the vertices of Γ have even valencies.

First, by way of terminology, define an *edge loop* in Γ to be a closed path in Γ consisting of a concatenation of edges that begin and end at the same vertex such that no edge is traversed twice. To construct an edge loop, start at a chosen vertex of Γ and start walking along an edge until its end, then continue from that end vertex along another edge, and so on, with it understood that no edge should be traversed more than once and that the walk should end when the starting vertex is reached. If all edges are not contained in this edge loop, then there is a vertex with at least two untouched incident edges, so a second edge loop can be constructed starting at that vertex that doesn't cross any edge from the first edge loop. If all edges are not contained in these two edge loops, then the process can be repeated because there will be a vertex with at least two uncrossed, incident edges. And then repeat again (and again) if needed until all edges have been crossed exactly once. Let $[L_1, \dots, L_p]$ denote the set of loops so constructed. These define distinct classes in $H_1(\Gamma; \mathbb{Z})$ (because they have distinct edges), so their union bounds a surface in $S^3 - \Gamma$. This surface will have intersection number 1 with each generator of $H_\Gamma^\perp$ since its boundary crosses each edge exactly once.

To complete the proof, suppose that Γ has a vertex with odd valency and there is a class obeying the conditions of the lemma. That class restricted to a small radius 2-sphere centered on the odd valency vertex would be non-trivial on the linking circles of the points where that 2-sphere intersects the edges of Γ . Thus, it will be non-trivial on a circle in that 2-sphere that encloses all of those intersection points (since there is an odd number). But that conclusion is nonsensical since that circle bounds a disk in the complement of those intersection points.

4. Some initial analysis

First, some notation: Here and subsequently, $\langle \cdot, \cdot \rangle$ is used to denote the inner product on T^*S^3 , on the spinor bundle for S^3 , on tensor bundles constructed from the latter, and in general, on any other vector bundle with a specified inner product. The Levi-Civita covariant derivative on sections of T^*S^3 , the corresponding covariant derivative on sections of the spinor

bundle, and on tensor bundles constructed from T^*S^3 and the spinor bundle will all be denoted by ∇ . Covariant derivatives are denoted by ∇ in what follows.

With this notation, let $\mathbb{V}$ denote a real or complex vector bundle over S^3 with a fiber metric and metric compatible connection. Let Γ denote a graph in S^3 with smoothly embedded edges and such that the edges abutting each vertex appear near the origin in a Gaussian coordinate chart centered at that vertex as smoothly embedded arcs that are asymptotically approaching the origin to a corresponding set of disjoint rays from the origin. Given $\mathbb{V}$, the tensor product $\mathbb{V} \otimes \mathcal{I}$ is a vector bundle over $S^3 - \Gamma$ with an inner product and compatible connection, and thus a covariant derivative (because, up to multiplication by ± 1 , $\mathbb{V}$ and $\mathbb{V} \otimes \mathcal{I}$ are canonically isomorphic over any open ball in $S^3 - \Gamma$). This covariant derivative is denoted by ∇ in what follows.

Granted now this covariant derivative: The positive square root of the functional

$$(4.1) \quad \psi \longmapsto \int_{S^3 - \Gamma} |\nabla \psi|^2$$

defines a norm on the space of smooth, compactly supported sections of $\mathbb{V} \otimes \mathcal{I}$ over $S^3 - \Gamma$.

Of particular note with regard to this norm: The non-triviality of $\mathcal{I}$ on the linking circles implies the inequality

$$(4.2) \quad \int_{S^3 - \Gamma} \frac{1}{\text{dist}(\cdot, \Gamma)^2} |\psi|^2 \leq c_0 \int_{S^3 - \Gamma} |\nabla \psi|^2,$$

where

$$\text{dist}(\cdot, \Gamma) := \min_{x \in \Gamma} \text{dist}(\cdot, x).$$

The norm on the space of compactly supported sections of $\mathbb{V} \otimes \mathcal{I}$ over $S^3 - \Gamma$ whose square is depicted in (4.1) is denoted by $\|\cdot\|_{\mathbb{H}}$. The completion of this space of compactly supported sections using the norm $\|\cdot\|_{\mathbb{H}}$ is a Hilbert space which is denoted by $\mathbb{H}$ in what follows.

Meanwhile, let $\mathbb{L}$ denote the Hilbert space completion of that same space of compactly supported sections of $\mathbb{V} \otimes \mathcal{I}$ over $S^3 - \Gamma$ using the norm whose square is the functional

$$(4.3) \quad \psi \longmapsto \int_{S^3 - \Gamma} |\psi|^2.$$

This norm on $\mathbb{L}$ is denoted by $\|\cdot\|_{\mathbb{L}}$.

Of interest in what follows are the functionals on $\mathbb{H}$ that can be written as

$$(4.4) \quad \psi \longmapsto \mathcal{E}(\psi) \equiv \int_{S^3 - \Gamma} \langle \nabla \psi, \nabla \psi \rangle + \int_{S^3 - \Gamma} \langle \psi, \mathcal{R} \psi \rangle,$$

with $\mathcal{R}$ here (and subsequently) denoting the extension to $\mathbb{V} \otimes \mathcal{I}$ of a symmetric (or Hermitian) endomorphism of the bundle $\mathbb{V}$. (The inequality in (4.2) implies that $\mathcal{E}$ is bounded by a c_0 -multiple of the square of the $\mathbb{H}$ -norm.)

Let E denote a given real number. A non-zero section ψ from $\mathbb{H}$ will be deemed to be a $(\nabla^\dagger \nabla + \mathcal{R})$ -eigensection with eigenvalue E in the event that the identity below holds whenever $\eta \in \mathbb{H}$:

$$(4.5) \quad \int_{S^3 - \Gamma} \langle \nabla \eta, \nabla \psi \rangle + \int_{S^3 - \Gamma} \langle \eta, \mathcal{R} \psi \rangle = E \int_{S^3 - \Gamma} \langle \eta, \psi \rangle.$$

Any given $(\nabla^\dagger \nabla + \mathcal{R})$ -eigensection is smooth on $S^3 - \Gamma$ and it obeys there the eigenvalue equation $(\nabla^\dagger \nabla + \mathcal{R})\psi = E\psi$. (The unwritten convention henceforth is that eigensections are normalized to have $\mathbb{L}$ -norm equal to 1.) The following lemma says in effect that $(\nabla^\dagger \nabla + \mathcal{R})$ -eigensections are ubiquitous.

LEMMA 4.1. The set of $(\nabla^\dagger \nabla + \mathcal{R})$ -eigenvalues is discrete, bounded from below and lacks accumulation point. Moreover, each such eigenvalue has finite multiplicity. Corresponding eigenvectors for distinct eigenvalues are $\mathbb{L}$ -orthogonal. Once an $\mathbb{L}$ -orthonormal basis is chosen for each eigenspace, the resulting collection of eigensections forms an orthonormal basis for $\mathbb{L}$.

This lemma is proved at the end of this section. The next lemma makes an assertion regarding the norm of a $(\nabla^\dagger \nabla + \mathcal{R})$ -eigensection near Γ .

LEMMA 4.2. The norm of any $(\nabla^\dagger \nabla + \mathcal{R})$ -eigensection from $\mathbb{H}$ extends across Γ as a Hölder continuous function that vanishes on Γ .

This lemma is proved in Section 8 using some key technical lemmas in Section 7. But note in this regard that there are steps in the proof of Section 7's technical lemma where Γ 's edges are assumed to be geodesic arcs. This assumption is obeyed in the cases from Theorem 1.2 since the graph Γ is the radial projection to S^3 of the edges and vertices of a polytope in $\mathbb{R}^4$. Lemma 4.2 is true without the simplifying assumption of geodesic arc edges, but the proof in the general case adds some length to the arguments and isn't given in this paper.

An example for Lemma 4.2 has $\mathbb{V}$ being the product $\mathbb{R}$ -bundle and $\mathcal{R} = 0$; thus $(\nabla^\dagger \nabla + \mathcal{R})$ in this case is (-1) times the standard spherical Laplacian on S^3 , but acting on sections of $\mathcal{I}$ over $S^3 - \Gamma$. This case is relevant for the construction of homogeneous, $\mathbb{Z}/2$ harmonic 1-forms and spinors on $\mathbb{R}^4$. A second relevant example is that where $\mathbb{V} = T^*S^3$ and where $\mathcal{R}$ is the Ricci curvature tensor in its incarnation as a symmetric endomorphism of T^*S^3 . This case is relevant for the construction of homogeneous, $\mathbb{Z}/2$ harmonic self-dual 2-forms on $\mathbb{R}^4$. An instance of a third example (which is discussed next) is also relevant to the $\mathbb{Z}/2$ harmonic, self-dual 2-form case.

Suppose now that $\mathbb{V}$ is also a Clifford module for T^*S^3 . This is to say that there is a monomorphism $\tau : T^*S^3 \rightarrow \text{End}(\mathbb{V})$ obeying the rules below:

$$\begin{aligned} & \bullet \tau(u) \tau(v) + \tau(v) \tau(u) = -2\langle u, v \rangle \text{ for all } u, v \in T^*S^3; \\ & \bullet \tau^\dagger = -\tau; \\ & \bullet \nabla \tau = \tau \nabla. \end{aligned} \tag{4.6}$$

Here, $\tau^\dagger$ signifies either the transpose or Hermitian conjugate of τ as the case may be. An additional assumption is (imposed henceforth) that this Clifford algebra action is compatible with the given connection on $\mathbb{V}$. Granted this Clifford module structure, there is a corresponding, first-order, elliptic and symmetric differential operator acting on sections of $\mathbb{V}$. This operator is denoted by $\mathcal{D}$; and having chosen an oriented, orthonormal frame for T^*S^3 , it can be written as

$$(4.7) \quad \mathcal{D} = \tau_a \nabla_a,$$

where $\{\tau_1, \tau_2, \tau_3\}$ are the images of the basis 1-forms via τ and where $\{\nabla_1, \nabla_2, \nabla_3\}$ are the directional covariant derivatives in the direction of the dual basis for TS^3 . This operator $\mathcal{D}$ also acts on the space of smooth, compactly supported sections of $\mathbb{V} \otimes \mathcal{I}$ over $S^3 - \Gamma$; and as such, it extends to define a bounded, linear map from $\mathbb{H}$ to $\mathbb{L}$. A given element (call it ψ) from $\mathbb{H}$ is said in what follows to be an eigensection for $\mathcal{D}$ when it is not identically zero and there exists a real number (denoted by λ) such that $\mathcal{D}\psi = \lambda\psi$ holds as an identity of sections of $\mathbb{L}$. (A convention henceforth: All $\mathcal{D}$ -eigensections have $\mathbb{L}$ -norm equal to 1.)

Because $\mathcal{D}$ is elliptic, its eigensections in $\mathbb{H}$ are smooth sections of $\mathbb{V} \otimes \mathcal{I}$ on $S^3 - \Gamma$. The lemma that follows makes an assertion about the norm of ψ .

LEMMA 4.3. The norm of an eigensection for $\mathcal{D}$ from $\mathbb{H}$ extends over Γ to define a Hölder continuous function on S^3 whose zero locus contains Γ .

Proof of Lemma 4.3. This is a corollary to Lemma 4.2 because an eigensection of $\mathcal{D}$ from $\mathbb{H}$ is also eigensection of $\mathcal{D}^2$, a $\mathcal{D}^2$ eigensection being an element ψ of $\mathbb{H}$ such that the identity

$$(4.8) \quad \int \langle \mathcal{D}\eta, \mathcal{D}\psi \rangle = \mathbf{E} \int \langle \eta, \psi \rangle$$

holds for all $\eta \in \mathbb{H}$ with $\mathbf{E}$ being independent of η . Meanwhile, a Bochner–Weitzenböck formula for $\mathcal{D}^2$ can be used to identify the left-hand side of (4.8) as an instance of the left-hand side of (4.5). (The integration by parts to go from (4.8) to (4.5) won't get hung up on Γ when both ψ and η are from $\mathbb{H}$.)

As explained directly, a $\mathcal{D}^2$ -eigensection from $\mathbb{H}$ can be used to construct a section from the Hilbert space $\mathbb{L}$ which is (formally) a $\mathcal{D}$ -eigensection. Indeed, supposing that ψ denotes a $\mathcal{D}^2$ eigensection from $\mathbb{H}$, then both the $+$ and $-$ versions of

$$(4.9) \quad \phi_\pm = \psi \pm \frac{1}{\sqrt{\mathbf{E}}} \mathcal{D}\psi$$

are in $\mathbb{L}$ and they obey the identity $\mathcal{D}\phi_\pm = \pm\sqrt{\mathbf{E}}\phi_\pm$ on $S^3 - \Gamma$. In particular, $\phi_\pm$ is a $\mathcal{D}$ -eigensection if it is in $\mathbb{H}$ and not identically zero. This may or may not be the case. Of course, ψ is in $\mathbb{H}$ by assumption, but $\mathcal{D}\psi$ need not be. But, if $\mathcal{D}\psi$ is indeed in $\mathbb{H}$, then one of ϕ_+ and ϕ_- can be used

to construct a harmonic, homogeneous $\mathbb{Z}/2$ object on $\mathbb{R}^4$ as explained in Section 2 (assuming that $\mathcal{D}$ comes via (2.12)).

To elaborate about $\mathcal{D}\psi$ being in $\mathbb{H}$: Not-with-standing the fact that $\mathcal{D}$ is a symmetric operator, and not-with-standing what Lemmas 4.1 and 4.2 imply for $\mathcal{D}^2$, the operator $\mathcal{D}$ is only semi-Fredholm; it has an infinite dimensional cokernel (see [21, 14, 1]). This is to say that there is an infinite-dimensional subspace in $\mathbb{L}$ that is annihilated by $\mathcal{D}$. An element in this subspace is smooth on $S^3 - \Gamma$, but it need not be in $\mathbb{H}$. Indeed, it need not be in $\mathbb{H}$ for the same reason that a solution from $\mathbb{L}$ to the equation $\mathcal{D}\psi = \lambda\psi$ need not be in $\mathbb{H}$, which is this: An integration by parts is needed to use the integrals of $|\mathcal{D}\psi|^2$ and $|\psi|^2$ to bound the integral of $|\nabla\psi|^2$, and this integration by parts on $S^3 - \Gamma$ gets hung up along Γ if ψ is in $\mathbb{L}$ but not in $\mathbb{H}$.

With Lemma 4.3 in hand, the question at hand is this: When does a $\mathcal{D}^2$ -eigensection from $\mathbb{H}$ give a $\mathcal{D}$ -eigensection from $\mathbb{H}$ via the formula in (4.9); thus when is $\mathcal{D}\psi$ in $\mathbb{H}$ also? The next lemma gives a sufficient condition for this.

LEMMA 4.4. In the context of (4.5), suppose that ψ denotes a $(\nabla^\dagger\nabla + \mathcal{R})$ -eigensection from $\mathbb{H}$. Then $\nabla\psi$ is in the $\mathbb{V}' \equiv \mathbb{V} \otimes T^*S^3$ version of $\mathbb{H}$ if, for every point q from any edge of Γ , there exist constants $c_q > 0$ and ρ_q such that the following is true: For any given $r \in (0, \rho_q]$, let ∂B_r denote the radius- r sphere centered at q and let $A(r)$ denote the area of ∂B_r (which is $4\pi \sin^2(r)$). Then

$$(4.10) \quad \frac{1}{A(r)} \int_{\partial B_r} |\psi|^2 \leq c_q r^2.$$

Section 8 explains how this lemma follows from those two technical lemmas in Section 7.

With regards to $\mathcal{D}\psi$ being in $\mathbb{H}$: It is possible for this to happen not-with-standing the fact that $\nabla\psi$ is not in $\mathbb{H}$. This possibility is exploited in the upcoming proof of Theorem 1.2's existence assertion for $\mathbb{Z}/2$ self-dual harmonic 2-forms.

To summarize: A $\mathcal{D}^2$ -eigensection ψ from $\mathbb{H}$ supplies a $\mathcal{D}$ -eigensection in $\mathbb{H}$ via (4.9) if the average of $|\psi|^2$ over the radius r sphere centered at any interior point of an edge of Γ is bounded by a constant multiple of r^2 . In any event, if $\mathcal{D}\psi$ is in $\mathbb{H}$, then the formula in (4.9) supplies an eigensection of $\mathcal{D}$ in $\mathbb{H}$. And, when $\mathcal{D}$ comes via (2.12), that eigensection for $\mathcal{D}$ from $\mathbb{H}$ can be used to construct a homogeneous, $\mathcal{D}$ -harmonic, $\mathbb{Z}/2$ object on $\mathbb{R}^4 - Z$.

Proof of Lemma 4.1. The proof follows the standard min-max variational route to obtain the $(\nabla^\dagger\nabla + \mathcal{R})$ -eigenvalues and eigenvectors as the respective critical values and critical points of the functional $\mathcal{E}$ from (4.4) subject to the constraint that $\|\psi\|_{\mathbb{L}} = 1$. A standard Sobolev inequality (applied to the L_1^2 function $|\psi|$ and ψ 's components with respect to a basis for $\mathbb{V}$) and the Rellich lemma can be brought to bear to implement the min-max procedure just as if no Γ were present (see for example [19]). (In this regard: The Rellich

lemma implies that a sequence in $\mathbb{H}$ with bounded $\mathbb{H}$ -norm has a Cauchy subsequence with respect to the $\mathbb{L}$ -norm.)

5. Equivariant analysis

The subsequent construction of $(\nabla^\dagger \nabla + \mathcal{R})$ -eigensections from $\mathbb{H}$ that obey the condition set forth in Lemma 4.4 exploits the isometric action on S^3 of certain finite groups (which is the S^3 -analog of the strategy employed in [26]). To set the stage, suppose for the moment that G is a finite group acting on S^3 via orientation-preserving isometries. Assume in addition that all elements in G map Γ to itself. The assumption in what follows is that G lifts to an action on $\mathcal{I}$. To elaborate: The pull-back of $\mathcal{I}$ by the action of any element from G is isometric to $\mathcal{I}$. Even so, that isometry will be ambiguous up to the action of $\{-1, 1\}$. With that understood, let $\tilde{G}$ denote the set of pairs of the form (g, δ) with $g \in G$ and δ being an isometry from $g^*\mathcal{I}$ to $\mathcal{I}$. This set $\tilde{G}$ is itself a group, it being a central extension of G by $\{-1, 1\}$; see Lemma 12.4. The action of the group G lifts to a fiberwise linear action on $\mathcal{I}$ if there exist, for each $g \in G$, an isomorphism $\delta_g : g^*\mathcal{I} \rightarrow \mathcal{I}$, and if these isomorphisms are compatible with the group law in the sense that

$$(5.1) \quad \delta_g \circ \delta_{g'} = \delta_{g'g}$$

for any two elements $g, g' \in G$. (This implies in particular that $\delta_{(\cdot)}$ for the identity element in G is the identity isomorphism from $\mathcal{I}$ to $\mathcal{I}$.)

Supposing that the G action on S^3 lifts to an action on $\mathcal{I}$ as described above, then G acts on the space of sections of $\mathcal{I}$ as follows: Let s denote a given section of $\mathcal{I}$ and let g denote a given element in G . Then g sends the section s to the section whose value at any given $x \in S^3 - \Gamma$ is $\delta_g s(g(x))$. That given section s is then G -invariant when $\delta_g s(g(x)) = s(x)$ for each $x \in S^3 - \Gamma$. A very simple example of how this works is given in the next paragraph. As explained in Section 11-17, other examples are the symmetry groups of the polytopes from Theorem 1.2.

By way of an example: Suppose that Γ 's intersection with a given small radius ball in S^3 is a subset of the set of $|x| = 1$ points on a plane through the origin in $\mathbb{R}^4$. No generality is lost by taking that plane to be the plane where $x_3 = x_4 = 0$. Let m denote an odd integer greater than 2 and let G denote the group $\mathbb{Z}/m\mathbb{Z}$ whose action on $\mathbb{R}^4$ and S^3 is generated by the $2\pi/m$ rotation of the coordinates (x_3, x_4) . To elaborate, let z denote the complex coordinate $x_3 + ix_4$. Then the generator of this group sends (x_1, x_2, z) to $(x_1, x_2, e^{2\pi i/m} z)$. Meanwhile, the line bundle $\mathcal{I}$ can be viewed as the intersection of the ball with the subset in $(\mathbb{R}^2 \times \mathbb{C}) \times \mathbb{C}$ consisting of the elements (x_1, x_2, z, u) with $z \neq 0$, with $x_1^2 + x_2^2 + |z|^2 = 1$ and such that

$$(5.2) \quad \bar{z} u = |z| \bar{u}.$$

To see about the lift of the group $\mathbb{Z}/m\mathbb{Z}$, view that group as the multiplicative group of m^{th} roots in $\mathbb{C}$ of the number 1. Let g denote the generator $e^{2\pi i/m}$. The pull-back via g of the line bundle $\mathcal{I}$ consists of the set

(x_1, x_2, x, u) with (x_1, x_2, z) as before but now with the constraint $\bar{z} e^{-2\pi i/m} u = |z| \bar{u}$. Define the element δ_g for the $\mathbb{Z}/m\mathbb{Z}$ generator g (which is $e^{2\pi i/m}$) according to the rule whereby any given u obeying the constraint $\bar{z} e^{-2\pi i/m} u = |z| \bar{u}$ is sent by δ_g to $u' \equiv -e^{-i\pi/m} u$ (which is in $\mathcal{I}$ since $\bar{z} u' = |z| \bar{u}'$). If m is odd, then the m^{th} power of the pair (g, δ_g) is equal to the identity in $\tilde{G}$; and so it defines a lift of the G action on $S^3 - \Gamma$ to a fiberwise linear action on $\mathcal{I}$. (There is no lift of the $\mathbb{Z}/m\mathbb{Z}$ to an action on $\mathcal{I}$ when m is even.)

Supposing now that G denotes a subgroup of $\text{SO}(4)$ (the isometry group of S^3) that maps Γ to itself, and that this group G does lift in the manner just described to a fiberwise linear action on $\mathcal{I}$, then there is an infinite dimensional subspace in $\mathbb{H}$ of G -invariant sections of $\mathcal{I}$. Indeed, if there exists just one non-trivial, G -invariant section, then there are infinitely many linearly independent ones because any such map can be multiplied by the pull-back to S of a G -invariant function on $S^3 - \Gamma$. Meanwhile, a G -invariant section can be obtained by taking a section with compact support in some small ball (disjoint from Γ) in a fundamental domain for G in S^3 and then averaging that section over its G -orbit.

In the preceding local example where G is $\mathbb{Z}/m\mathbb{Z}$ with m an odd, positive integer: Suppose that k is a positive integer and a is a complex number, then

$$(5.3) \quad (x_1, x_2, z, u) = \left(x_1, x_2, z, a \frac{1}{\sqrt{|z|}} z^k + \bar{a} \bar{z}^{k-1} \sqrt{|z|} \right)$$

is a section of $\mathcal{I}$; but it is a G -invariant section of $\mathcal{I}$ if and only if m is an odd integer and k is congruent modulo m to $\frac{1}{2}(m+1)$. To put the section above in a more familiar form, take $\sqrt{z}$ to be defined on the complement of the negative x_3 -axis. Then $u_0 \equiv \sqrt{z} \frac{1}{\sqrt{|z|}}$ is a unit length section of $\mathcal{I}$ on the complement of that axis (it satisfies (5.2) there). The section u in (5.3) is a real multiple of this basis section, that multiple being the real part of $az^{k-1/2}$.

The following lemma has more to say about G -invariant sections of $\mathcal{I}$.

LEMMA 5.1. Let G denote a finite subgroup of $\text{SO}(4)$ mapping Γ to itself and lifting to a fiberwise linear action on $\mathcal{I}$. Granted this assumption, then there exists an infinite (but countable) set of G -invariant eigensections in $\mathbb{H}$ for the standard Laplacian; and these span the subspace in $\mathbb{L}$ of G -invariant sections of $\mathcal{I}$.

Lemma 5.1 follows as an instance of the next lemma which makes an analogous assertion for $(\nabla^\dagger \nabla + \mathcal{R})$ -eigenvectors. The upcoming assertion requires *a priori* that G satisfies two constraints: The first requires the G -action on $S^3 - \Gamma$ lifts to a fiberwise, isometric action on the total space of both $\mathbb{V}$ and $\mathbb{V} \otimes \mathcal{I}$. The second is that the operator $(\nabla^\dagger \nabla + \mathcal{R})$ when acting on the sections of $\mathbb{V} \otimes \mathcal{I}$ is G -equivariant.

LEMMA 5.2. Suppose that G is a finite subgroup of $\mathrm{SO}(4)$ that maps Γ to itself. Assuming the preceding two constraints on the group G , then there exists an infinite (but countable) set of G -invariant $(\nabla^\dagger \nabla + \mathcal{R})$ -eigensections in $\mathbb{H}$; and these span the subspace in $\mathbb{L}$ of G -invariant sections of $\mathbb{V} \otimes \mathcal{I}$.

Proof of Lemmas 5.1 and 5.2. Use the same min-max eigenvalue characterization as for the proof of Lemma 4.1 after restricting the domain of $\mathcal{E}$ to the subspace in $\mathbb{H}$ of G -invariant sections (with $\mathbb{L}$ -norm equal to 1). Doing that leads to an $\mathbb{L}$ -orthonormal set of G -invariant sections from $\mathbb{H}$ with the following two properties:

- Any G -invariant section from $\mathbb{L}$ is a linear combination of sections from this set.
- If ψ is from this set, and if η is any G -invariant eigensection from $\mathbb{H}$, then (4.5) holds with E being independent of η . (5.4)

The two lemmas follow from (5.4) if it is the case that (4.5) holds with the same number E when η is *not* G -invariant. But this follows because any given section of $\mathbb{L}$ can be written as the sum of a G -invariant part (obtained by averaging over the G -action) and a part that is orthogonal to the subspace of G -invariant elements. In particular, if η is orthogonal to the G -invariant subspace in $\mathbb{L}$, then it will be $\mathbb{H}$ -orthogonal to that subspace; and this implies that both sides of (4.5) are zero for any such element η .

What follows describes a relevant example for Lemma 5.2. Take Γ as in the previous example (the intersection of the $x_3 = x_4 = 0$ plane with a ball in S^3 centered on that plane). Likewise, take $G = \mathbb{Z}/m\mathbb{Z}$ with m being an odd integer. Introduce coordinates z (a complex coordinate) and t (a real coordinate) on that ball with $z = x_3 + ix_4$ as before and with t defined by writing $x_1 + ix_2$ as $(1 - |z|^2)^{1/2} e^{it}$ and such that $t = 0$ at the ball's center. Take the vector space $\mathbb{V}$ to be T^*S^3 . The differential forms dt and the real and imaginary parts of dz span T^*S^3 on that ball.

The $\mathbb{Z}/m\mathbb{Z}$ action on S^3 lifts canonically to an action on T^*S^3 via pull-back; and the generator $g = e^{2\pi i/m}$ acts according to the rule whereby $g^*dt = dt$ and $g^*dz = e^{2\pi i/m} dz$. As noted previously, the $\mathbb{Z}/m\mathbb{Z}$ action lifts to an action on $\mathcal{I}$ if m is odd. Thus, if m is odd, then the action lifts to an action on $T^*S^3 \otimes \mathcal{I}$. As for invariant sections: A section of $T^*S^3 \otimes \mathcal{I}$ can be written as

$$(5.5) \quad w_t dt + \frac{1}{2} (w_1 - iw_2) dz + \frac{1}{2} (w_1 + iw_2) d\bar{z},$$

where w_t , w_1 and w_2 are sections of $\mathcal{I}$ (they must pointwise obey (5.2) where $z \neq 0$). If this section is sent to itself via the $\mathbb{Z}/m\mathbb{Z}$ action, then this must be the case for w_t , whereas w_1 and w_2 must be such that

$$(5.6) \quad \delta_g(g^*(w_1 - iw_2)) e^{2\pi i/m} = w_1 - iw_2.$$

To find sections w_1 and w_2 obeying (5.6), consider the case where

(5.7)

$$w_1 = a_1 \frac{1}{\sqrt{|z|}} z^k + \bar{a}_1 \bar{z}^{k-1} \sqrt{|z|} \quad \text{and} \quad w_2 = a_2 \frac{1}{\sqrt{|z|}} z^{k'} + \bar{a}_2 \bar{z}^{k'-1} \sqrt{|z|},$$

where a_1 and a_2 are complex numbers, and where k and k' are positive integers. This choice for w_1 and w_2 obeys (5.6) if and only if $k = k'$ and a_1 and a_2 are such that

$$\begin{aligned} & -e^{\pi i(2k-1)/m} e^{2\pi i/m} (a_1 - ia_2) = (a_1 - ia_2) \\ (5.8) \quad & \text{and} \quad -e^{-\pi i(2k-1)/m} e^{2\pi i/m} (\bar{a}_1 - i\bar{a}_2) = (\bar{a}_1 - i\bar{a}_2). \end{aligned}$$

These preceding two equations for a_1 and a_2 are solvable if and only if either $k \equiv \frac{1}{2}(m-1) \pmod{m}$ or $k \equiv \frac{1}{2}(m+3) \pmod{m}$. In the first instance, $a_1 = -ia_2$; and in the second instance, $a_1 = ia_2$. This is to say the following:

- When $k \equiv \frac{1}{2}(m-1) \pmod{m}$, then $w_1 - iw_2 = -2ia_2 \frac{1}{\sqrt{|z|}} z^k$.
 - When $k \equiv \frac{1}{2}(m+3) \pmod{m}$, then $w_1 - iw_2 = -2i\bar{a}_2 \bar{z}^{k-1} \sqrt{|z|}$.
- (5.9)

When considering derivatives of sections of $\mathcal{I}$: Remember that derivatives of sections of $\mathcal{I}$ are defined using isometries between $\mathcal{I}$ and the product $\mathbb{R}$ -bundle on balls in $S^3 - \Gamma$. However, when $\mathcal{I}$ is depicted as in (5.2) and sections of $\mathcal{I}$ appear as a map from $S^3 - \Gamma$ to $\mathbb{C}$ obeying (5.2), then the derivative of that section appears as the covariant derivative of the corresponding map to $\mathbb{C}$ with respect to a particular flat connection on $S^3 - \Gamma$. To be explicit, if u denotes the map to $\mathbb{C}$ obeying (5.2), then the relevant covariant derivative is

$$(5.10) \quad \nabla u = du - \frac{i}{2} d\theta u$$

with d being the usual derivative on maps to $\mathbb{C}$ and with θ denoting the argument of the coordinate z (write z in polar form as $|z|e^{i\theta}$). For example, $u_0 \equiv \sqrt{z} \frac{1}{\sqrt{|z|}}$ is a unit length, *covariantly constant* section of $\mathcal{I}$ on the complement of the negative real axis in $\mathbb{C}$ (it satisfies (5.2) there and it is annihilated by the covariant derivative in (5.10)). What follows is a consequence:

The $\mathcal{I}$ -valued 1-form $a = \frac{1}{2}(w_1 - iw_2)dz + \frac{1}{2}(w_1 + iw_2)d\bar{z}$ from (5.6) is closed when $k = k'$ with both being $\frac{1}{2}(m-1) \pmod{m}$. This is to say that $da = 0$. Moreover, although d^*a is not zero, its norm is bounded by a constant multiple of $|a|$.

(5.11)

As explained subsequently, this last observation plays a key role in the upcoming proof of the assertions in Theorem 1.2 regarding self-dual, $\mathbb{Z}/2$ harmonic 2-forms. By way of comparison, if $k = k'$ and both equal $\frac{1}{2}(m +$

3) mod (m) , then da is not zero; in fact, the norm of da and that of d^*a are both $O(|z|^{-1}|a|)$.

6. Hölder continuity across the interior of the edges of Γ

This section states and then proves preliminary results that are needed for the proof of Lemma 4.2 and for later use. These preliminary results are summarized by the upcoming Lemmas 6.1 and 6.2. To set the background for what is to come: A graph in S^3 has been specified (it is again denoted by Γ); and it is assumed implicitly that this graph has even valencies so that there is a corresponding real line bundle $\mathcal{I}$ on $S^3 - \Gamma$. A vector bundle $\mathbb{V}$ on S^3 has also been specified along with fiber metric and metric compatible connection. As before, $\mathbb{H}$ denotes the Hilbert space completion of the space of smooth sections of $\mathbb{V} \otimes \mathcal{I}$ with compact support in $S^3 - \Gamma$ using the norm whose square is depicted in (4.1). Also as before, use $\mathbb{L}$ to denote the completion of that same space of sections using the norm whose square is depicted in (4.3). Supposing that $\mathcal{R}$ denotes a given endomorphism of $\mathbb{V}$ (as in (4.4)), the promised Lemmas 6.1 and 6.2 concern $(\nabla^\dagger \nabla + \mathcal{R})$ -eigensections from the Hilbert space $\mathbb{H}$.

By way of a look ahead: These lemmas concern $(\nabla^\dagger \nabla + \mathcal{R})$ -eigensections near interior points of edges from the graph Γ . Lemma 6.1 concerns the pointwise norm of an eigensection from $\mathbb{H}$ near an interior point of an edge; and Lemma 6.2 concerns the directional derivative of an eigensection from $\mathbb{H}$ in the direction tangent to that nearby edge. Both Lemma 6.1 and 6.2 assume implicitly that the edges of Γ are geodesic arcs. (Both lemmas are true when the geodesic arc assumption is not made, but the proofs in that generality are not needed for this paper.)

To set the notation for what follows momentarily: Supposing that p denotes a given point in $S^3 - \Gamma$, the notation has r_p denoting the distance from p to Γ . When p is in Γ but not a vertex of Γ , the number r_p denotes the distance from p to the nearest vertex of Γ . When p is a vertex of Γ , then r_p denotes the distance from p to the nearest of the edges of Γ that don't have an endpoint at p . An important geometric point to keep in mind regarding the various $p \in \Gamma$ versions of r_p is the following:

There exists a number to be denoted by c (it is greater than 100) such that when p is a point from Γ , then the only edges of Γ that intersect the radius $\frac{1}{c}r_p$ ball centered at p are the edges that contain p , the intersection of that radius $\frac{1}{c}r_p$ ball centered at p with Γ being a union of half-open arcs with p being their common starting point.

(6.1)

Since the geometry is a bit subtle, the subsequent digression to describe it might prove illuminating. The digression refers to the following diagram, which sketches a 3-dimensional cartoon version of part of a graph in S^3 , as illustrated in Fig. 1.

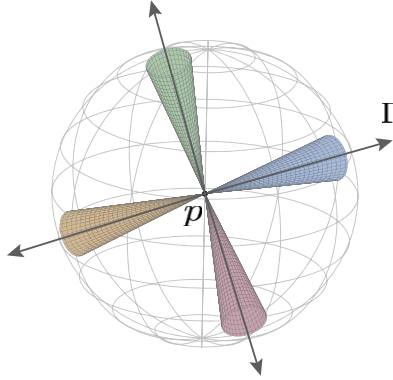


FIGURE 1. Local geometry near a vertex $p \in V(\Gamma)$. The sphere represents $\partial B_{r_p}(p)$ and the colored conical regions represent uniformly separated distance cones around the incident edges. The picture shows the valency-4 case.

What should first catch your eye in this sketch is a sphere centered at a vertex of Γ . Supposing that p denotes that vertex, then the radius of this sphere is the corresponding r_p , where r_p is the distance from p to the nearest edge of Γ that is not incident to p . The interior of this sphere corresponds to the ball in S^3 with distance less than r_p from p . This vertex p has valency 4; its incident edges are the black lines that have p as an endpoint. The colored cones indicate the regions in S^3 whose closest point in Γ lies on the corresponding edge, with the distance from a given point to that closest point q bounded by $\frac{1}{8c}r_q$. Note that the solid-angle cross-sections of these cones are very nearly $\frac{1}{8c}$. The idea is that, after shrinking to the ball of radius $c^{-1}r_p$, only the incident edges remain in view and these conical regions are uniformly separated.

What follows is the first of the promised lemmas from this section.

LEMMA 6.1. Let ψ denote a $(\nabla^\dagger \nabla + \mathcal{R})$ -eigensection from $\mathbb{H}$ with $\mathbb{L}$ -norm equal to 1. If q signifies an interior point of an edge from Γ , then the function $|\psi|$ on $S^3 - \Gamma$ extends as a continuous function across Γ in the radius $\frac{1}{4c}r_q$ ball centered at q ; and $|\psi|$ on this ball obeys

$$(6.2) \quad |\psi| \leq \kappa \operatorname{dist}(\cdot, \Gamma)^{1/2}$$

with κ being independent of the chosen point q .

This lemma can be proved using the technology that was introduced in [14], and also [8]. A proof using a different approach is given below.

The second lemma introduces additional notation: To define this new notation, let q denote a point in the interior of an edge of Γ . The part of Γ in the radius $\frac{1}{4c}r_q$ ball centered at q is then a geodesic arc which is an arc in a great circle. Denote that great circle by γ . Fix Euclidean coordinates $x = (x_1, x_2, x_3, x_4)$ for $\mathbb{R}^4$ so that γ appears as the radius 1 circle in the

x_1, x_2 plane, i.e., $x_3 = x_4 = 0$. The $\mathbb{R}^4$ vector field $x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}$ is tangent to S^3 along S^3 (which is the $|x| = 1$ locus in $\mathbb{R}^4$) and it is tangent to γ along γ . In addition, this vector field along S^3 generates an isometry of S^3 which maps γ to itself as the group of constant rotations of γ . With respect to the coordinates (z, t) from Section 5 for a ball centered at a point in γ , the vector field $x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}$ is just $\frac{\partial}{\partial t}$. (By way of a reminder: The complex coordinate z is the restriction of $x_3 + ix_4$ to S^3 , and t comes by writing the complex coordinate $x_1 + ix_2$ on that ball as $(1 - |z|^2)^{1/2} e^{it}$ with $t = 0$ at ball's center point.)

More notation for the second lemma: Given q as above, let χ_q denote a favorite non-increasing, smooth function with values in $[0, 1]$ that is equal to 1 where the distance to q is less than $\frac{1}{8c}r_q$, equal to 0 where the distance to q is greater than $\frac{1}{4c}r_q$, and such that $|d\chi_q| \leq 100c \frac{1}{r_q}$.

LEMMA 6.2. Let ψ denote a $(\nabla^\dagger \nabla + \mathcal{R})$ -eigensection from $\mathbb{H}$ with $\mathbb{L}$ -norm equal to 1. Restricting to the radius $\frac{1}{8c}r_q$ ball centered at q , let $\nabla_t \psi$ denote the directional derivative of ψ along the vector field $\frac{\partial}{\partial t}$. Then $\chi_q \nabla_t \psi$ is in the Hilbert space $\mathbb{H}$; and its $\mathbb{H}$ -norm is at most $\kappa \frac{1}{\sqrt{r_q}}$ with κ being independent of q .

The rest of this subsection has five parts with Part 4 containing the proof of Lemma 6.1 and Part 5 containing the proof of Lemma 6.2. Part 1 states some direct *a priori* bounds for $(\nabla^\dagger \nabla + \mathcal{R})$ -eigensections that are used later. The intervening Parts 2 and 3 then introduce and analyze a Dirichlet Green's function for the operator $\nabla^\dagger \nabla$ acting on sections of $\mathbb{V} \otimes \mathcal{I}$ over a given small radius ball centered on an interior point of an edge of Γ . This Green's function is then used in Part 4 to analyze the behavior of a given $(\nabla^\dagger \nabla + \mathcal{R})$ -eigensection from $\mathbb{H}$ near that interior point of the given edge of Γ . Using the Green's function in this way requires some specific bounds for that Green's function near Γ ; and those bounds are relatively straightforward to come by when Γ is a geodesic arc because symmetry considerations can be exploited. That is why the geodesic arc assumption is made in Lemma 6.1. As for the geodesic arcs in Lemma 6.2: The proof of that lemma uses the bounds from Lemma 6.1, but it also uses the fact that there are isometries of S^3 that restrict to any given geodesic (when parametrized by arc-length) as a rigid rotation.

Part 1: Let ψ denote a $(\nabla^\dagger \nabla + \mathcal{R})$ -eigensection from $\mathbb{H}$ with $\mathbb{L}$ -norm equal to 1, and let $\mathbf{E}$ denote its eigenvalue. The first observations in this part of Section 6 are the inequalities below for the $\mathbb{L}$ -norm of $\nabla \psi$, for a certain weighted version of the $\mathbb{L}$ -norm of ψ , and for the pointwise norm of $|\psi|$:

- $\int |\nabla \psi|^2 \leq c_0(1 + \mathbf{E}).$
- $\int \frac{1}{\text{dist}(\cdot, \Gamma)^2} |\psi|^2 \leq c_0(1 + \mathbf{E}).$

- $|\psi|$ extends to S^3 as an L^∞ function obeying $|\psi| \leq c_0(1 + \mathbf{E})$.
- For any point $x \in S^3$: $\int \frac{1}{\text{dist}(\cdot; x)} |\nabla \psi|^2 \leq c_0(1 + \mathbf{E})$. (6.3)

By way of a proof: The first of these inequalities follows directly from (4.5) by taking η to be the section ψ . The second inequality in (6.3) follows from (4.2) and the first inequality. The third and fourth inequalities in (6.3) are proved using the fact that ψ obeys the equation below on $S^3 - \Gamma$:

$$(6.4) \quad \nabla^\dagger \nabla \psi + \mathcal{R}\psi = \mathbf{E}\psi.$$

To say more: The identity in (6.4) implies that $|\psi|^2$ obeys the differential inequality below:

$$(6.5) \quad -\frac{1}{2} \Delta^\perp |\psi|^2 + |\nabla \psi|^2 \leq c_0(1 + \mathbf{E}) |\psi|^2.$$

Now, having chosen a point $x \in S^3 - \Gamma$, multiply both sides of (6.5)'s inequality by the Green's function with pole at x for the operator $-\Delta^\perp + 1$ acting on $\mathbb{R}$ -valued functions on S^3 . Integrate the result over S^3 and then integrate by parts to see that

$$(6.6) \quad |\psi|^2(x) + \int \frac{1}{\text{dist}(\cdot; x)} |\nabla \psi|^2 \leq c_0(1 + \mathbf{E}) \int \frac{1}{\text{dist}(\cdot; x)} |\psi|^2.$$

The integration by parts doesn't get hung up on Γ even when x is from Γ because ψ is from $\mathbb{H}$. This can be justified as done in [23] using a sequence of bounded approximations to the Green's function, for example the sequence $\{G/(1 + \frac{1}{n}G)\}_{n=1,2,\dots}$ with G denoting just here the $-\Delta^\perp + 1$ Green's function. Use of the Cauchy-Schwarz inequality with (6.6) and the first bullet in (6.3), and with a dimension-3 Sobolev inequality leads to the assertions of the third and fourth assertions in (6.3). (That Sobolev inequality says in effect that the fourth root of the integral of the fourth power of an L_1^2 function over S^3 is no larger than c_0 times that function's L_1^2 norm. The function in this case is $|\psi|$. Remember in this regard that Kato's inequality bounds at each point the norm of $d|\psi|$ by the norm of $\nabla \psi$.)

Part 2: Suppose in what follows that $\mathbf{B}$ is a ball in S^3 with radius at most $\frac{\pi}{4}$ (and thus it sits inside a hemisphere). The set of elements in $\mathbb{H}$ that vanish on $S^3 - \mathbf{B}$ is the completion using the $\mathbb{H}$ -norm of the space of smooth sections of $\mathbb{V} \otimes \mathcal{I}$ that have compact support on $\mathbf{B} - (\mathbf{B} \cap \Gamma)$. This subspace is denoted by $\mathbb{H}_{\mathbf{B}}$. The completion of the space of smooth sections of $\mathbb{V} \otimes \mathcal{I}$ over $\mathbf{B} - (\mathbf{B} \cap \Gamma)$ using the $\mathbb{L}$ -norm is denoted by $\mathbb{L}_{\mathbf{B}}$; it is a subspace of the Hilbert space $\mathbb{L}$.

The lemma below is the centerpiece of Part 2.

LEMMA 6.3. There exists $\kappa > 1$ with the following significance: Let $\mathbf{B}$ denote an open ball in S^3 with radius $\rho < \frac{\pi}{4}$. Given $\varsigma \in \mathbb{L}_{\mathbf{B}}$, there is a unique element $\psi \in \mathbb{H}_{\mathbf{B}}$ which obeys

$$(6.7) \quad \nabla^\dagger \nabla \psi = \varsigma \quad \text{on } \mathbf{B} - (\mathbf{B} \cap \Gamma),$$

and is such that the identity

$$(6.8) \quad \int_{\mathbf{B}} \langle \nabla \eta, \nabla \psi \rangle = \int_{\mathbf{B}} \langle \eta, \varsigma \rangle$$

holds for all elements $\eta \in \mathbb{H}_{\mathbf{B}}$. This ψ obeys the bound

$$(6.9) \quad \int_{\mathbf{B}} \left(|\nabla \psi|^2 + \frac{1}{\rho^2} |\psi|^2 \right) \leq \kappa \rho^2 \int_{\mathbf{B}} |\varsigma|^2.$$

Proof of Lemma 6.3. The desired section ψ can be obtained by minimizing the functional on $\mathbb{H}_{\mathbf{B}}$ given by the rule

$$(6.10) \quad \psi \longmapsto \frac{1}{2} \int_{\mathbf{B}} |\nabla \psi|^2 - \int_{\mathbf{B}} \langle \psi, \varsigma \rangle.$$

The existence of a unique minimizer in $\mathbb{H}_{\mathbf{B}}$ can be readily proved using (4.2) with input from the fact that the inequality

$$(6.11) \quad \int_{\mathbf{B}} |\nabla \psi|^2 \geq c_0^{-1} \rho^{-2} \int_{\mathbf{B}} |\psi|^2$$

holds for all $\psi \in \mathbb{H}_{\mathbf{B}}$. (This follows from Kato's inequality $|\nabla \psi| \geq |d|\psi|$ and the fact that (6.11) holds when ψ is an ordinary function with compact support in $\mathbf{B}$.) The existence of the minimizer also invokes the Rellich lemma to conclude that bounded $\mathbb{H}_{\mathbf{B}}$ -norm sequences have $\mathbb{L}_{\mathbf{B}}$ -norm Cauchy subsequences.

Part 3: Let $\mathbf{B}$ denote an (open) ball of radius less than $\frac{\pi}{4}$ centered at an interior point of an edge in Γ whose closure has distance at least twice $\mathbf{B}$'s radius from any other edge in Γ . This step states and then proves a lemma regarding a Dirichlet Green's function for the operator $\nabla^\dagger \nabla$. By way of terminology, a Green's function for $\nabla^\dagger \nabla$ is an example of a singular integral operator, that being an object which is defined for the purposes of the upcoming lemma as follows: Let $(\mathbb{V} \otimes \mathcal{I})_R$ and $(\mathbb{V} \otimes \mathcal{I})_L$ denote the respective pull-backs to

$$(6.12) \quad (\mathbf{B} - (\mathbf{B} \cap \Gamma)) \times (\mathbf{B} - (\mathbf{B} \cap \Gamma))$$

of the bundle $\mathbb{V} \otimes \mathcal{I}$ by the projections to the right- and left-hand factors of $\mathbf{B} - (\mathbf{B} \cap \Gamma)$. A singular integral operator is a smooth section of the vector bundle

$$\mathrm{Hom}((\mathbb{V} \otimes \mathcal{I})_R, (\mathbb{V} \otimes \mathcal{I})_L)$$

over the complement of the diagonal in the product space depicted in (6.12), which is characterized by the following property: Let ς denote any given smooth section of $\mathbb{V} \otimes \mathcal{I}$ over $\mathbf{B} - (\mathbf{B} \cap \Gamma)$. The section of $\mathbb{V} \otimes \mathcal{I}$ over $\mathbf{B} - (\mathbf{B} \cap \Gamma)$ that is obtained by first acting by that section of $\mathrm{Hom}((\mathbb{V} \otimes \mathcal{I})_R, (\mathbb{V} \otimes \mathcal{I})_L)$ on the pull-back of ς to the product space in (6.12) via the right-hand projection and then integrating the result over that same right-hand factor yields a section of $\mathbb{V} \otimes \mathcal{I}$ in the Hilbert space $\mathbb{H}_{\mathbf{B}}$.

The lemma below is the centerpiece for Part 3.

LEMMA 6.4. Suppose in what follows that Γ is a graph in S^3 whose vertices have even valency and whose edges are geodesic arcs. Given this graph Γ , there exists $\kappa > 1$ with the following significance: If the radius of a ball $B \subset S^3$ is sufficiently small (less than $\frac{1}{\kappa}$), then there is a linear operator (denoted by G_B) mapping $\mathbb{L}_B$ to $\mathbb{H}_B$ with operator norm at most κ , and which has the properties listed in the subsequent bullets.

- If $\varsigma \in \mathbb{L}_B$, then $\psi = G_B(\varsigma)$ is the section from Lemma 6.3 that obeys $\nabla^\dagger \nabla \psi = \varsigma$ on $B - (B \cap \Gamma)$, and is such that

$$(6.13) \quad \int_B \langle \nabla \eta, \nabla \psi \rangle = \int_B \langle \eta, \varsigma \rangle \quad \text{for all } \eta \in \mathbb{H}_B.$$

- The operator G_B has a unique representation as a Green's function, and in particular, as a singular integral operator for which the bounds below hold for any given pair of distinct points (x, y) in $B - (B \cap \Gamma)$:

$$\text{a) } |G_B(x, y)| \leq \frac{\kappa}{\text{dist}(x, y)}.$$

$$\text{b) } \text{If } \text{dist}(x, \Gamma) \leq \frac{1}{2} \text{dist}(y, \Gamma), \text{ then } |G_B(x, y)| \leq \kappa \frac{\text{dist}(x, \Gamma)^{1/2}}{\text{dist}(y, \Gamma)^{3/2}}.$$

- c) *For any fixed $y \in B - (B \cap \Gamma)$, the function $|\nabla G_B(\cdot, y)|$ has finite integral on $B - (B \cap \Gamma)$; this integral being at most κ . Meanwhile, the integral of $|\nabla G_B(\cdot, y)|^2$ over the complement in $B - (B \cap \Gamma)$ of any open ball centered at y is finite, that integral being at most $\frac{\kappa}{\delta}$, with δ denoting the radius of that ball.*

Proof of Lemma 6.4. The first bullet simply restates conclusions from Lemma 6.3. As for the second bullet: The existence of the Green's function can be proved using standard arguments (see, e.g., [10]) given that $\mathcal{I}$ is isometric to the product $\mathbb{R}$ -bundle on any ball in $B - (B \cap \Gamma)$. In any event, some remarks about a construction of the Green's function are given below in Step 3 of the upcoming six-step proof.

With regards to the assumption that the edges of Γ are geodesic arcs: This is the case for the graphs that are defined by the radial projections to S^3 of the polytopes from Theorem 1.2. In any event, the proofs that follow of Items a) and c) of Lemma 6.4's second bullet don't make the geodesic arc assumption; and Item b) is true without that assumption, but some extra (relatively straightforward) steps are needed to deal honestly with that level of generality.

The six steps of the proof explain how to obtain the bounds that are asserted in Items a), b), and c) of the lemma's second bullet. What follows directly are two facts about the Dirichlet Green's function for the Laplacian acting on $\mathbb{R}$ -valued functions on the ball B . This is a function on the complement of the diagonal in $B \times B$ that is harmonic with respect to x for fixed y and vice versa. (It is symmetric with respect to the map sending (x, y) to (y, x) .) Of particular note are the properties listed below:

- *This Green's function is positive, and it is bounded by $\frac{c_0}{\text{dist}(\cdot, y)}$.*

- The norm of this Green's function's differential (which is a 1-form on the complement of the diagonal in $B \times B$) is bounded by $\frac{c_0}{\text{dist}(x, y)^2}$.

(6.14)

With regards to the steps in the proof: Step 1 proves Item a) of the lemma's second bullet and Step 2 proves Item c) of that bullet. Step 3 is an aside regarding the existence of the Green's function G_B . Steps 4 and 5 prove Item b) of the lemma's second bullet for the case when $\mathbb{V}$ is the product $\mathbb{R}$ -bundle and thus $\mathbb{V} \otimes \mathcal{I} \cong \mathcal{I}$ and when the edges of the graph are geodesic arcs. The sixth step proves Item b) for any given $\mathbb{V}$ under that same assumption about the edges.

Step 1: Fix a point $y \in B - (B \cap \Gamma)$ and then an isometric identification between $\mathcal{I}|_y$ and $\mathbb{R}$. Transport this product structure at y along the radial geodesics from y to identify $\mathcal{I}$ with the product $\mathbb{R}$ -bundle on any ball in $B - (B \cap \Gamma)$ centered at y . Given any positive number r that is less than one-fourth of the distance from y to Γ , let $u_{y,r}$ denote the characteristic function for the radius r ball centered at y . Fix a unit length section of $\mathbb{V}$ over this radius r ball to be denoted by t_y and use the product structure for $\mathcal{I}$ on this radius r ball to view $u_{y,r}t_y$ as a section of $\mathbb{V} \otimes \mathcal{I}$ from the Hilbert space $\mathbb{L}_B$. Let $\psi_{y,r}$ denote the corresponding section from $\mathbb{H}_B$ given by Lemma 6.3. The identity

$$(6.15) \quad -\Delta^\perp \psi_{y,r} = u_{y,r}t_y$$

implies a differential inequality below for the norm of $\psi_{y,r}$ which is that

$$(6.16) \quad -\Delta^\perp |\psi_{y,r}| \leq \frac{1}{r^3} u_{y,r}.$$

Granted this inequality, then the comparison/maximum principle can be brought to bear to compare $|\psi_{y,r}|$ with the function $f_{y,r}$ on B that obeys the equation $-\Delta^\perp f_{y,r} = u_{y,r}$ and vanishes on B 's boundary. The comparison finds that

$$(6.17) \quad |\psi_{y,r}(x)| \leq c_0 \frac{1}{r^3} \int_B \frac{1}{\text{dist}(\cdot; x)} u_{y,r}.$$

(The function on the right comes via the top bullet of (6.14) since $f_{y,r}$ at any given point x is given by the integral over B of the product of $u_{y,r}$ with Dirichlet Green's function on B with pole at x .) In particular, if $\text{dist}(x, y) < 2r$, then the bound in (6.17) implies in turn that

$$(6.18) \quad |\psi_{y,r}(x)| \leq c_0 \frac{1}{\text{dist}(x, y)}.$$

Letting $r \rightarrow 0$ gives the bound in Item a) of the lemma's second bullet.

Step 2: Take that same $\psi_{y,r}$ with r still positive. Fix $\delta \in (0, \frac{1}{100} \text{dist}(y, \Gamma))$ and let $\beta_{y,\delta}$ denote a smooth, non-negative function mapping to $[0, 1]$ which is zero on the radius $\frac{1}{2}\delta$ ball centered at y and one on the complement of the radius δ ball centered at y . This function can and should be chosen so that

$|d\beta_{y,\delta}| \leq c_0 \delta^{-1}$. Now take the inner product of both sides of the equation $-\Delta^\perp \psi_{y,r} = u_{y,r} t_y$ with $\beta_{y,\delta}^2 \psi_{y,r}$ and integrate the result over $B - (B \cap \Gamma)$. Supposing that $r < \frac{1}{4}\delta$, then an integration by parts (which is allowed because $\psi_{y,r} \in \mathbb{H}_B$) together with (6.18)'s bound leads to this inequality:

$$(6.19) \quad \int_B |\nabla(\beta_{y,\delta} \psi_{y,r})|^2 \leq c_0 \frac{1}{\delta}.$$

Noting that the right-hand side is independent of r , taking $r \rightarrow 0$ proves the second assertion of Item c) of the lemma's second bullet.

To prove that $|\nabla G_B(\cdot, y)|$ has finite integral, set $\delta_0 = \frac{1}{1000} \text{dist}(y, \Gamma)$, and for each positive integer n , let δ_n denote $\frac{1}{2^n} \delta_0$. It follows from (6.18) and (6.19) that the integral of $|\nabla \psi_{y,r}|$ over the complement of the radius δ_0 ball centered at y is at most $c_0 \delta_0^{-1/2}$. By the same token, for any positive integer n , the integral of $|\nabla \psi_{y,r}|$ over the spherical annulus centered at y with inner radius δ_n and outer radius δ_{n-1} is at most $c_0 \delta_n^{-1/2} (\delta_{n-1})^{3/2}$, which is at most $c_0 2^{-n} \delta_0$. Thus, the sum of these annular contributions to the integral of $|\nabla G_B(\cdot, y)|$ is finite and at most $c_0 \delta_0$, as

$$(6.20) \quad \sum_{n=1}^{\infty} \int_{A_n} |\nabla \psi_{y,r}| \leq c_0 \delta_0 \sum_{n=1}^{\infty} 2^{-n} = c_0 \delta_0.$$

Step 3: This step is an aside for a remark regarding the construction of the Green's function G_B . The remark is that the existence of the $r \rightarrow 0$ limits for (6.18) and (6.19) can be used to prove the existence of G_B . This is because sections of the form $u_{y,r} t_y$ are dense in $\mathbb{L}_B$; and hence any given section ς of $\mathbb{L}_B$ can be approximated to any given accuracy (as measured by the $\mathbb{L}_B$ -norm) by a finite linear combination of these sorts of sections with disjoint support and with any given positive upper bound for the r -values. As a consequence, the corresponding ψ from Lemma 6.3 can be approximated to any given accuracy (as measured by the $\mathbb{H}_B$ -norm) by the corresponding finite linear combination of the $\psi_{y,r}$ sections.

Step 4: This step and the next step prove Item b) of the lemma's second bullet for the case when $\mathbb{V}$ is the product $\mathbb{R}$ -bundle and thus $\mathbb{V} \otimes \mathcal{I} \cong \mathcal{I}$. This step, the next one, and Step 6 all assume that the edges of Γ are geodesic arcs.

To describe G_B when $\mathbb{V}$ is the product $\mathbb{R}$ -bundle, it proves convenient to introduce the Gaussian normal coordinates (denoted by x_1, x_2, x_3) centered at the point q so as to identify the ball B with the ball of radius r_B in $\mathbb{R}^3$ centered at the origin and with the intersection of B with the edge containing q being the $-r_B < x_3 < r_B$ part of the x_3 -axis. In these coordinates, the spherical Riemannian metric is invariant with respect to rotations about the origin, and both the metric and B 's part of the chosen edge from Γ are invariant with respect to rotations of the (x_1, x_2) plane. This being the case, it is sufficient for what follows to take y to be a point where x_2 is zero and

x_1 is positive. Let Π denote the part of B where $x_1 \leq 0$ and $x_2 = 0$ (this is B 's intersection with a half-plane in $\mathbb{R}^3$).

As explained in the following, the endomorphism $G_B(\cdot, y)$ in this case (when $V = \mathbb{R}$) must vanish on Π . To see why, note first that $\mathcal{I}$ can be identified with the product line bundle on $B - \Pi$ with the identification such that $G_B > 0$ near y . Now take $u_{y,r}$ as before and let $g_{y,r}$ denote the function on $B - \Pi$ that obeys the conditions below:

$$\begin{aligned} \bullet \quad & -\Delta^\perp g_{y,r} = u_{y,r}, \\ \bullet \quad & g_{y,r}|_{\partial B} = 0 \quad \text{and} \quad g_{y,r}|_\Pi = 0. \end{aligned} \tag{6.21}$$

This function is positive on $B - \Pi$ and, by symmetry (the symmetry of the Laplacian when x_2 is replaced by $-x_2$), the x_2 -derivative of $g_{y,r}$ at any given point on the $x_1 < 0$ part of Π coming from positive x_2 side must be -1 times the x_2 -derivative at that same point coming from the negative x_2 side. It follows from this that $g_{y,r}$ can be viewed as a smooth section of $\mathcal{I}$ on $B - (B \cap \Gamma)$. And, as such, this section of $\mathcal{I}$ obeys the same equation and same Dirichlet boundary conditions on ∂B as $G_B(\varsigma)$ with $\varsigma = u_{y,r}t_y$ where t_y here denotes the unit length section of $\mathcal{I}$ on $B - \Pi$ that is mapped to $1 \in \mathbb{R}$ by the aforementioned isometry between $\mathcal{I}$ and the product $\mathbb{R}$ -bundle. Thus, $g_{y,r} = G_B(\varsigma)$. Since this holds for all r , it follows that

$$(6.22) \quad \lim_{r \rightarrow 0} \frac{1}{r^3} g_{y,r} = G_B(\cdot, y).$$

Thus, $G_B(\cdot, y)$ vanishes on Π as claimed (and nowhere else in $B - \Gamma$). (The preceding argument for the vanishing of $G_B(\cdot, y)$ on Π is a simplistic version of a very clever construction in [9].)

Step 5: Continue here with the assumptions and notation of the previous step. Let $B' \subset B$ denote the ball centered at the origin in $\mathbb{R}^3$ with radius equal to $\frac{1}{100} \text{dist}(y, \Gamma)$. Let ρ and θ denote the standard polar coordinates on the (x_1, x_2) plane in $\mathbb{R}^3$ with the $\theta = 0$ ray being the positive x_1 axis. Let $u = \sqrt{\rho} \cos(\frac{1}{2}\theta)$. This is a positive function on $B - \Pi$ that extends continuously over Π so as to be zero on Π . It is also harmonic on $B - \Pi$ with respect to the Euclidean metric on B . With respect to the metric from the sphere (which is conformal to the Euclidean metric), this function is such that

$$(6.23) \quad \Delta^\perp u \leq 0.$$

Hold onto u for the moment. In the meantime, fix a very small but positive number to be denoted by ε and let w_ε denote the $\Delta^\perp$ -harmonic, $\mathbb{R}$ -valued function on the spherical annulus in B where ρ is between ε and $\frac{1}{100} \text{dist}(y, \Gamma)$ subject to the boundary conditions that w_ε vanish both on ∂B and where $\rho = \frac{1}{100} \text{dist}(y, \Gamma)$, and that w_ε is equal 1 where $\rho = \varepsilon$. This function w is non-negative and it obeys the bound

$$(6.24) \quad w_\varepsilon(x) \leq c_0 \ln\left(\frac{\text{dist}(y, \Gamma)}{100\rho}\right) \bigg/ \ln\left(\frac{\text{dist}(y, \Gamma)}{100\varepsilon}\right).$$

(The bounds in (6.14) can be used to prove the preceding bound by bounding w_ε by a weighted integral of (6.14)'s Green function.)

By virtue of what is said above about u and w , and by virtue of the fact that $G_B(x, y)$ is bounded by $c_0 \frac{1}{\text{dist}(x, y)}$, and by virtue of the fact that $G_B(\cdot, y)$ vanishes on Π , there exists $c \in (1, c_0)$ such that the function

$$(6.25) \quad h_\varepsilon = c \frac{1}{\text{dist}(y, \Gamma)} (u + w_\varepsilon).$$

is larger than $G_B(\cdot, y)$ on the boundary of the complement in $B' - (B' \cap \Pi)$ to the cylinder where $\rho = \varepsilon$. This being the case, the maximum/comparison principle can be brought to bear using (6.23) and the fact that w_ε is harmonic to see that $|G_B(\cdot, y)| \leq c_0 h_\varepsilon$ on the complement of that same cylinder in $B' - (B' \cap \Pi)$. The claim in Item b) of the second bullet of the lemma for the case where $\mathbb{V}$ is the product $\mathbb{R}$ -bundle and Γ 's edges are geodesic arcs follows directly from the latter fact by taking $\varepsilon \rightarrow 0$ while noting that the right-hand side of (6.24) limits to zero at fixed ρ as $\varepsilon \rightarrow 0$.

Step 6: This step finishes the proof of Item b) of the lemma's second bullet under the assumption that the edges of Γ are geodesic arcs. To this end, first fix an identification between $\mathcal{I}|_y$ and the product $\mathbb{R}$ -bundle as before. Next, choose an orthonormal basis for $\mathbb{V}$ at the point q and parallel transport that chosen basis using the metric's Levi-Civita connection along the radial geodesic arcs from q so as to obtain a basis for $\mathbb{V}|_B$. Denote this basis as $\{t_1, \dots, t_{\dim(\mathbb{V})}\}$. Now use this basis to write $G_B(x, y)$ as

$$(6.26) \quad \sum G_{\alpha\beta}(x, y) t_\alpha|_x \otimes t_\beta|_y$$

with it understood that the summation is over the labels α, β , which come from the index set for $\mathbb{V}$'s orthonormal basis. Each $G_{\alpha\beta}(\cdot, y)$ is a section of $\mathcal{I}$ over B that vanishes on ∂B . As such, it obeys an equation on the complement of y that has the schematic form

$$(6.27) \quad -\Delta^\perp G_{\alpha\beta} + Q_{\alpha\gamma} \cdot \nabla G_{\gamma\beta} + P_{\alpha\gamma} G_{\gamma\beta} = 0$$

where the repeated index (γ in this case) is implicitly summed over that same index set and where y is fixed with differentiation on the x coordinate. (The tensor-valued functions Q and P are smooth on a neighborhood of B in S^3 with uniform derivative bounds to any given order.) It follows as a consequence of (6.27) that $G_{\alpha\beta}(x, y)$ with $x \neq y$ can be written using the Green's function for $-\Delta^\perp$ acting on sections of the line bundle $\mathcal{I}$ (here denoted by G) as

$$(6.28) \quad G_{\alpha\beta}(x, y) = G(x, y) \delta_{\alpha\beta} - \int_B \left(G(x, \cdot) (Q_{\alpha\gamma} \cdot \nabla G_{\gamma\beta} + P_{\alpha\gamma} G_{\gamma\beta}) \right) \Big|_{(\cdot, y)}.$$

Hold on to this identity for the moment.

Let $B' \subset B$ again denote the ball centered at the origin in $\mathbb{R}^3$ with radius equal to $\frac{1}{100} \text{dist}(y, \Gamma)$. Let B'' denote the ball concentric to B with radius

$\frac{1}{500} \text{dist}(y, \Gamma)$. Supposing that x is from B'' , an appeal to the $\mathbb{V} = \mathbb{R}$ case of Item b) of the lemma's second bullet, proved in Steps 4–5 implies the inequality below:

$$(6.29) \quad |G_{\alpha\beta}(x, y)| \leq c_0 \frac{\text{dist}(x, \Gamma)^{1/2}}{\text{dist}(y, \Gamma)^{3/2}} \left(1 + \int_{B-B''} (|\nabla G_B|_{(\cdot, y)} + |G_B|_{(\cdot, y)}) \right) \\ + c_0 \int_{B''} |G|_{(x, \cdot)} (|\nabla G_B|_{(\cdot, y)} + |G_B|_{(\cdot, y)}).$$

As for these integrals: An appeal to Items a) and c) of the lemma's second bullet for the case when $\mathbb{V} = \mathbb{R}$ (and thus $\mathbb{V} \otimes \mathcal{I} \cong \mathcal{I}$) supply the bounds asserted below:

$$\begin{aligned} \bullet \int_{B-B''} (|\nabla G_B|_{(\cdot, y)} + |G_B|_{(\cdot, y)}) &\leq c_0, \\ \bullet \int_{B'} |G|_{(x, \cdot)} (|\nabla G_B|_{(\cdot, y)} + |G_B|_{(\cdot, y)}) &\leq c_0 \text{dist}(x, \Gamma)^{1/2}. \end{aligned} \quad (6.30)$$

And, using these bounds in (6.29) finishes the proof of Item b) of the lemma's second bullet for the case when Γ 's edges are geodesic arcs.

Part 4: This part of the section proves Lemma 6.1.

Proof of Lemma 6.1: To start the proof, let q denote a given point in the interior of a given edge of Γ . Use q to define the number r_q as done at the beginning of this section, and let B denote the ball of radius $\frac{1}{2c} r_q$ centered at q . Use χ_q to denote (for the purposes of this proof) a smooth function with values in $[0, 1]$ that equals 1 in the radius $\frac{3}{8c} r_q$ ball centered at q and equals 0 near the boundary of B . This function can and should be constructed so that the norm of its differential and Hessian are bounded respectively by $c_0 \frac{1}{r_q}$ and by $c_0 \frac{1}{r_q^2}$.

Let ψ_q denote $\chi_q \psi$, this being a section of $\mathbb{V} \otimes \mathcal{I}$ on $B - (B \cap \Gamma)$ with compact support on B . It is in the Hilbert space $\mathbb{H}_B$ (this subspace of $\mathbb{H}$ is defined at the start of Part 2). By virtue of (6.4), the section ψ_q obeys an equation that can be written schematically as

$$(6.31) \quad \nabla^\dagger \nabla \psi_q = \mathbf{E} \psi_q - \mathcal{R} \cdot \psi_q + 2 d\chi_q \cdot \nabla \psi + (\Delta^\perp \chi_q) \psi.$$

where $d\chi_q \cdot \nabla \psi$ denotes the directional covariant derivative of ψ along the gradient of the function χ_q .

With (6.31) in hand, use the Green's function G_B from Lemma 6.3 to depict ψ_q at a point x in B as done below:

$$(6.32) \quad \psi_q(x) = \int_B G_B(x, \cdot) \left(\mathbf{E} \chi_q \psi - \mathcal{R} \cdot (\chi_q \psi) + 2 d\chi_q \cdot \nabla \psi + (\Delta^\perp \chi_q) \psi \right).$$

To exploit the preceding identity, suppose that x is in the radius $\frac{1}{4c} r_q$ ball centered at q . This is inside the set where $\psi_q = \psi$. Moreover, the distance from x to the support of $|d\chi_q|$ is no less than $\frac{1}{8c} r_q$. (This last fact is important when it comes to analyzing the contributions to $|\psi(x)|$ by the right-most two terms in (6.32).) Introducing r to denote the distance from x to Γ , let B' denote the part of B where the distance to Γ is less than $4r$, and let B'' denote

the complementary part of B . By virtue of the inequality from Item a) of the second bullet in Lemma 6.4, the contribution to $|\psi(x)|$ from B' is at most

$$(6.33) \quad c_0 \left(\int_{B'} \frac{1}{\text{dist}(x, \cdot)^2} \right)^{1/2} \left(\left(1 + E + \frac{1}{r_q} \right) \|\psi\|_{\mathbb{H}} + \frac{1}{r_q^2} \|\psi\|_{\mathbb{L}} \right).$$

This in turn is bounded courtesy of (6.3) by $c_0 \left(1 + \frac{1}{r_q} \right) (1 + E) \sqrt{r}$. Meanwhile, by virtue of the inequality from Item b) of the second bullet in Lemma 6.4 and (6.3), the contribution to $|\psi(x)|$ from B'' is at most

$$(6.34) \quad c_0 \sqrt{r} \left(\int_{B''} \frac{1}{\text{dist}(\Gamma, \cdot)^{3/2}} (1 + E) |\psi| + \frac{1}{r_q^{1/2}} (1 + E) \right).$$

To elaborate: The contribution to (6.34) from the terms in (6.32)'s integral with derivatives on χ_q is accounted for by the right-most term in (6.34) (the term with the factor of $r_q^{-1/2}$). The contribution to (6.34) from (6.32)'s integral from its two $\chi_q \psi$ terms is accounted for by the term with the explicit integral in (6.34). With regards to that integral: The third and fourth bullets in (6.3) bound that integral by $c_0(1 + E)$. As a consequence, the expression in (6.34) is bounded in turn by $c_0 \sqrt{r} \left(1 + \frac{1}{\sqrt{r_q}} \right) (1 + E)$.

Since $r = \text{dist}(x, \Gamma)$, the $c_0 \sqrt{r} \left(1 + \frac{1}{r_q} \right) (1 + E)$ bound for both the B' and B'' contributions to the integral on the right-hand side of (6.34) prove Lemma 6.1.

Part 5: This last part of the section justifies the assertion of Lemma 6.2.

Proof of Lemma 6.2: When ε is from the interval $(0, \frac{1}{256c} r_q)$, let u_ε denote the isometry of S^3 that is obtained by integrating the vector field $\frac{\partial}{\partial t}$ for time ε . Then, let ψ_ε denote the pull-back of ψ by this isometry. This ψ_ε is a section of the pull-back bundle $u_\varepsilon^*(\mathbb{V} \otimes \mathcal{I})$. Use parallel transport along the integral curves of $\frac{\partial}{\partial t}$ to identify this bundle over the radius $\frac{1}{4c} r_q$ ball centered at q with $\mathbb{V} \otimes \mathcal{I}$. Both ψ_ε and ψ are sections of bundle $\mathbb{V} \otimes \mathcal{I}$ over that same ball, and as such, they both obey $(\nabla^\dagger \nabla + \mathcal{R})(\cdot) = E(\cdot)$. Also, the integrals of $|\nabla \psi|^2$ and $|\nabla \psi_\varepsilon|^2$ over that ball are both bounded by $c_0 \rho^{-2}$ times the integral of $|\psi|^2$ over B . That in turn is bounded by $c_0 \rho^3$ (see Lemma 6.1). Thus, the integrals over B of both $|\nabla \psi|^2$ and $|\nabla \psi_\varepsilon|^2$ are bounded by $c_0 \rho$.

The plan for what follows is to examine the $\varepsilon \rightarrow 0$ behavior of $(\psi_\varepsilon - \psi)$ inside the ball B . This section is denoted by η_ε in what follows; and it obeys $(\nabla^\dagger \nabla + \mathcal{R})\eta_\varepsilon = E\eta_\varepsilon$. To localize the subsequent analysis to B , reintroduce the function χ_q and take the inner product of the equation $(\nabla^\dagger \nabla + \mathcal{R})\eta_\varepsilon = E\eta_\varepsilon$ with $\chi_q^2 \eta_\varepsilon$; then integrate the resulting function over B (where it has compact support). An instance of integration by parts with (6.3) leads to

the inequality below

$$(6.35) \quad \int_{\mathbf{B}} |\nabla(\chi_q \eta_\varepsilon)|^2 \leq c_0 r_q^{-2} \int_{\mathbf{B}} |\eta_\varepsilon|^2.$$

The key observation now is that $\varepsilon^{-1}(\psi_\varepsilon - \psi)$ converges in the $\mathbb{L}$ -topology to $\nabla_t \psi$. Hence, the product of ε^{-2} times the right-hand side of (6.35) converges as $\varepsilon \rightarrow 0$ to $c_0 r_q^{-2}$ times the integral over $\mathbf{B}$ of $|\nabla_t \psi|^2$. It follows as a consequence this (and the dominated convergence theorem) and that $\chi_q \nabla_t \psi$ is a limit of a uniformly $\mathbb{H}$ -norm bounded sequence of elements in $\mathbb{H}$ thus that it is in $\mathbb{H}$ also. Moreover, given the bound in the top bullet of (6.3), taking $\varepsilon \rightarrow 0$ in (6.35) leads to an *a priori* bound for the $\mathbb{H}$ -norm of $\chi_q \nabla_t \psi$, this being $c_0 r_q^{-1}$ times the square root of the integral over $\mathbf{B}$ of $|\nabla_t \psi|^2$. This last fact implies Lemma 6.2's asserted bound because the square root of the integral over $\mathbf{B}$ of $|\nabla_t \psi|^2$ has a $c_0 r_q^{1/2}$ upper bound courtesy of (6.3) and Lemma 6.1.

7. Almgren's frequency function

This section presents various technical observations and constructions that are also needed for the proofs of Lemmas 4.2 and 4.4. Some of these technical observations and constructions are used again in later parts of the proof of Theorem 1.2.

Assume at the outset that a graph in S^3 has been specified (it is again denoted by Γ), and that it has even valencies so that there is a corresponding real line bundle $\mathcal{I}$ on $S^3 - \Gamma$. Likewise, assume that a vector bundle $\mathbb{V}$ on S^3 has been specified with fiber metric and metric-compatible connection. These are fixed throughout the subsequent discussion. As before, $\mathbb{H}$ denotes the Hilbert space completion of the space of smooth sections of $\mathbb{V} \otimes \mathcal{I}$ with compact support in $S^3 - \Gamma$ using the norm whose square is depicted in (4.1). Also, as before, use $\mathbb{L}$ to denote the completion of that same space of sections using the norm whose square is depicted in (4.3). Supposing that $\mathcal{R}$ denotes a given endomorphism of $\mathbb{V}$ (as in (4.4)), the constructions in what follow concern $(\nabla^\dagger \nabla + \mathcal{R})$ -eigensections from the Hilbert space $\mathbb{H}$.

What follows next is a digression to further set the stage and notation for the definition of the frequency function. To start the digression, suppose that ψ again denotes a $(\nabla^\dagger \nabla + \mathcal{R})$ -eigensection from $\mathbb{H}$ and let $\mathbf{E}$ again denote its eigenvalue. For any given point $p \in S^3$ and $r \in (0, \frac{1}{1000}\pi]$, use $\mathbf{B}_r$ to denote the radius r ball centered at p and use $\partial \mathbf{B}_r$ to denote $\mathbf{B}_r$'s boundary. With p fixed in advance, a positive function on $(0, \frac{1}{1000}\pi]$ to be denoted by $\mathbf{K}$ is defined by the rule below, whereby

$$(7.1) \quad \mathbf{K}^2(r) = \frac{1}{A(r)} \int_{\partial \mathbf{B}_r} |\psi|^2,$$

where $A(r) = 4\pi \sin^2(r)$ is the area $\partial \mathbf{B}_r$. Thus, $\mathbf{K}^2$ is the average of $|\psi|^2$ over the radius r sphere centered at p . This function $\mathbf{K}$ is a differentiable function

on $(0, \frac{1}{1000}\pi]$ as an integration by parts (which is allowed if $\psi \in \mathbb{H}$) can be used to write its derivative as below:

$$(7.2) \quad K \frac{dK}{dr} = \frac{1}{A(r)} \int_{B_r} (|\nabla \psi|^2 + \langle \psi, \mathcal{R}\psi \rangle - E|\psi|^2).$$

As for the $r \rightarrow 0$ limit of K : The function K extends continuously to $r = 0$ except possibly in the cases where the chosen point p is in Γ . In any event, if $|\psi|$ is continuous at a chosen point p , then $K_p(0) = |\psi|(p)$.

Fix a point $p \in S^3$. Almgren's frequency function is defined from the corresponding version of K : This frequency function is denoted by N ; it is defined (initially) on $(0, \frac{1}{1000}\pi]$ by writing the derivative of the function K as

$$(7.3) \quad \frac{dK}{dr} = \frac{N(r)}{r} K(r).$$

It follows from (7.2) that N is a continuous function on $(0, \frac{1}{1000}\pi]$ that can be depicted as the two bullets below:

$$\begin{aligned} \bullet N(r) &= \frac{r}{2AK^2} \int_{\partial B_r} (\langle \psi, \nabla_r \psi \rangle + \langle \nabla_r \psi, \psi \rangle), \\ \bullet N(r) &= \frac{r}{AK^2} \int_{B_r} (|\nabla \psi|^2 + \langle \psi, \mathcal{R}\psi \rangle - E|\psi|^2). \end{aligned} \quad (7.4)$$

The first line is obtained by directly differentiating K and the second follows from the first using the divergence theorem (integration by parts). What follows are two lemmas that state key properties of the function N and then a third lemma that states a key property of the function K .

LEMMA 7.1. *There exists $\kappa > 1$ such that the assertions below hold for any $(\nabla^\dagger \nabla + \mathcal{R})$ -eigensection from $\mathbb{H}$. Fix a point in S^3 to define the functions K and then N from the given eigensection. The chosen point's version of function N obeys*

- $N(r) \geq -\kappa(1 + E)r^2$.
- The function N has a unique $r \rightarrow 0$ limit; and with regards to that limit,
 - (a) $\lim_{r \rightarrow 0} N(r) = 0$ if ψ is non-zero at the chosen point.
 - (b) $\lim_{r \rightarrow 0} N(r) > 0$ if the chosen point is in a vertex of Γ .
 - (c) $\lim_{r \rightarrow 0} N(r) \geq \frac{1}{2}$ if the chosen point is in the interior of an edge of Γ .
- Suppose that $\{p_n\} \subset S^3$ converges to a chosen point in the interior of an edge of Γ . Then that chosen point's version of $N(0)$ is no smaller than the $\limsup$ of the respective p_n versions of $N(0)$.
- Suppose that p denotes a vertex of Γ and suppose that the versions of $N(0)$ for points on the interiors of the edges of Γ in a neighborhood of p are all strictly greater than $\frac{1}{2}$. Then p 's version of $N(0)$ is not less than the $\limsup$ of the versions of $N(0)$ that are defined by any sequence of edge points from Γ that converges to p .

Fix a basepoint $p \in S^3$. Let $N_p(r)$ denote the frequency function defined by (7.3), with the integrals taken over the geodesic ball $B_r(p)$ and its

boundary $\partial B_r(p)$. By Lemma 7.1, $N_p(r)$ has a limit as $r \rightarrow 0$; we set

$$N_p(0) := \lim_{r \rightarrow 0} N_p(r).$$

When p is fixed, we often suppress the subscript and write $N(r)$ and $N(0)$. We write $N_{(\cdot)}(0)$ for the resulting function $p \mapsto N_p(0)$.

The proof of this lemma is given momentarily.

The second lemma refers to numbers from the set $\{r_p : p \in S^3\}$ and the positive number c that are described just prior to (6.1), and then in (6.1) and Fig. 1. This is the promised second lemma about the function N .

LEMMA 7.2. Let ψ denote a $(\nabla^\dagger \nabla + \mathcal{R})$ -eigensection from $\mathbb{H}$ with $\mathbb{L}$ -norm equal to 1. There exists $\kappa > 0$ with the following significance: Supposing that p denotes a given point in S^3 and supposing that $r \in (0, \frac{1}{8c} r_p)$, then the corresponding version of N is almost everywhere differentiable on $(0, \frac{1}{8c} r_p)$; and its derivative (where N is differentiable) obeys the inequality below:

$$\frac{dN}{dr} \geq \frac{r}{AK^2} \int_{\partial B_r} \left| \nabla_r \psi - \frac{N}{r} \psi \right|^2 - \kappa(1 + N)r.$$

Moreover, if $\rho > 0$, and if $N_{(\cdot)}(0) > \frac{1}{2}$ at each point from the part of Γ inside the radius ρ ball centered at a vertex of Γ , and if p has distance at most $\frac{1}{2}\rho$ from that vertex, then the preceding inequality holds at any differentiable value for the function N from the whole interval $(0, \kappa^{-1}\rho)$.

The proof of this lemma is also given momentarily.

The next lemma states some facts about the function K .

LEMMA 7.3. There exists $\kappa > 1$ with the following significance: Let ψ denote a $(\nabla^\dagger \nabla + \mathcal{R})$ -eigensection from $\mathbb{H}$ with $\mathbb{L}$ -norm equal to 1. Fix a point from Γ (denoted by p) to define the corresponding functions K and N . If $r \in (0, \frac{1}{4c} r_p)$, then

$$K(r) \leq \kappa z_p r^{N(0)}$$

with z_p being independent of r . As a consequence,

$$\int_{\partial B_r} |\psi|^2 \leq 4\pi \kappa z_p^2 r^{2+2N(0)} \quad \text{and} \quad \int_{B_r} |\psi|^2 \leq \frac{4\pi}{3} \kappa z_p^2 r^{3+2N(0)}.$$

With regards to z_p : Suppose that p is in the interior of an edge from Γ . Let q denote the nearest vertex in Γ to p and let $N_q(0)$ denote q 's version of the number $N(0)$. Define $\bar{N}$ to be the maximum of 0 and $N(0) - N_q(0)$. Then, $z_p \leq \kappa r_p^{-\bar{N}}$.

Proof of Lemma 7.3. Fix $p \in \Gamma$ and integrate (7.3) between r and $\frac{1}{4c} r_p$. Regarding that integral, Lemma 7.2 leads to the inequality

$$(7.5) \quad K(r) \leq c_0 K\left(\frac{1}{4c} r_p\right) \left(\frac{1}{r_p}\right)^{N(0)} r^{N(0)}.$$

Let z_p now denote $K(\frac{1}{4c}r_p) \left(\frac{1}{r_p}\right)^{N(0)}$. The lemma's integral inequalities follow from (7.5).

To see about a bound for z_p : Integrate (7.3) again to see that $K(\rho) \geq c_0^{-1} z_p \rho^{N(0)}$ for any given $\rho \in [\frac{1}{4c}r_p, \frac{1}{2c}r_p]$. This implies in turn that

$$(7.6) \quad \int_{B_{\frac{r_p}{2c}} - B_{\frac{r_p}{4c}}} |\psi|^2 \geq c_0^{-1} z_p^2 r_p^{3+2N(0)}.$$

If $\frac{1}{2c}r_p$ is greater than $\frac{1}{1000c}$ times the shortest edge length, then (7.6) will run afoul of the $\|\psi\|_{\mathbb{L}} = 1$ constraint in the event that $z_p \geq c_0$.

To see about the case when r_p is relatively small, let q denote the closest vertex to p . For any given positive number ρ , let B'_ρ denote the radius ρ ball centered at q . It then follows from (7.6) using (7.3) that the inequality below holds for $\rho = \frac{1}{c}r_p$:

$$(7.7) \quad \int_{B'_\rho} |\psi|^2 \geq c_0^{-1} z_p^2 \rho^{3+2N(0)}.$$

Let $N'(0)$ denote the vertex q 's version of the number $N(0)$. Granted (7.7), use (7.3) with q replacing p to see that

$$(7.8) \quad \int_{B'_{r_q}} |\psi|^2 \geq c_0^{-1} z_p^2 \left(\frac{r_q}{r_p}\right)^{3+2N'(0)} r_p^{3+2N(0)}.$$

Since $r_q \geq c_0^{-1}$, this bound runs afoul of the $\|\psi\|_{\mathbb{L}} = 1$ constraint if $z_p > c_0/r_p^{\bar{N}}$ with $\bar{N}$ denoting the maximum of 0 and $N(0) - N'(0)$.

Proof of Lemma 7.1. The assertion made by the lemma's first bullet follows from the uniform L^∞ bound for $|\psi|$ from the third bullet of (6.3). Since other assertions of the lemma invoke the conclusions of Lemma 7.2, assume that lemma is true for now.

With regards to the assertion of the second bullet of Lemma 7.1 regarding a $r \rightarrow 0$ limit for N : Lemma 7.2 says in effect that the derivative of N is no smaller than $-c_0(1+N)r$ on an interval of the form $(0, \frac{1}{2c}r_p)$ or $(0, c_0^{-1}\rho)$ as the case may be. Integration of this derivative inequality implies that a function of the form

$$(7.9) \quad r \longmapsto e^{c_0 r} N(r) + c_0 r^2$$

is non-decreasing on that same interval. This non-decreasing property implies in turn the existence of a unique $r \rightarrow 0$ limit for N since N is bounded from below.

Granted that there is a unique $r \rightarrow 0$ limit for N , then Item a) of Lemma 7.1's second bullet follows by integrating (7.3). Given what is said by Lemma 6.1, a completely analogous application of (7.3) leads directly to the conclusions of Item c) of Lemma 7.1. To see about Item b) of that bullet, note that the bound in (4.2) holds with the integrals on both sides restricted to the ball of radius r centered on the given vertex of Γ . Given the

formula in (7.4), that bound leads to the lower bound below, which holds when r is sufficiently small:

$$(7.10) \quad N(r) \geq c_0^{-1} \frac{r}{AK^2} \int_0^r K^2(s) ds.$$

If the $r \rightarrow 0$ limit of $N(r)$ is greater than 1, then there is nothing to prove. If that limit is less than 1, then (7.3) implies that $K(s) \geq \frac{1}{2}K(r)$ for s between $\frac{1}{2}r$ and r ; and then (7.10) leads directly to a c_0^{-1} lower bound for $N(r)$ when r is small.

With regards to the third bullet of Lemma 7.1: Let p denote the limit point inside that edge of Γ . Fix a small value for r and let B_r denote the ball of radius r centered at p . Given a point q from that same edge of Γ , let B'_r denote the ball of radius r centered at q . A standard measure-theoretic fact is this: Given $\varepsilon > 0$ but very small, there exists $\delta > 0$ such that if q is in that same edge and $\text{dist}(p, q) < \delta$, then

$$(7.11) \quad \begin{aligned} & \bullet \quad \int_{B_r} (|\nabla\psi|^2 + \langle\psi, \mathcal{R}\psi\rangle - E|\psi|^2) \text{ and} \\ & \int_{B'_r} (|\nabla\psi|^2 + \langle\psi, \mathcal{R}\psi\rangle - E|\psi|^2) \text{ differ by at most } \varepsilon, \\ & \bullet \quad \frac{1}{A(r)} \int_{\partial B_r} |\psi|^2 \text{ and } \frac{1}{A(r)} \int_{\partial B'_r} |\psi|^2 \text{ differ by at most } \varepsilon. \end{aligned}$$

For that same q as above, let $N'(r)$ denote q 's version of N . It follows from the formula in the second bullet of (7.4) using (7.11) that if δ is sufficiently small (with r fixed), then $N(r)$ and $N'(r)$ differ by at most ε . Keeping this in mind, suppose that r is small enough so that $N(r)$ and $N(0)$ also differ by at most ε . Meanwhile, $N'(r)$ is no smaller than $e^{-c_0\varepsilon}N'(0) - c_0\varepsilon$ if r is small and q is very close to p . (See what is said regarding (7.9) in the proof above for the lemma's second bullet. Note in this regard, that if p is in the interior of an edge, and if q is very close to p , then the distance from p to q will be very much smaller than $\frac{1}{2c}r_q$.) Thus, if r small to begin with and then q is sufficiently close to p , then it follows that $N(0)$ is no smaller than $e^{-c_0\varepsilon}N'(0) - c_0\varepsilon$. Since ε can be any small positive number, this last inequality establishes the assertion from the lemma's third bullet.

With regards to the fourth bullet's assertion: A key input for the argument given above for the third bullet was the assertion that the function depicted in (7.9) is non-decreasing as long as r is less than $\frac{1}{2c}r_p$. That upper bound for r can be taken to be independent of the point p (thus c_0^{-1}) for any point sufficiently close to the given vertex on an edge that ends at that vertex if the conditions for the fourth bullet of Lemma 7.1 are met. Granted this uniform upper bound for r at those same edge points, then the argument used for the third bullet of Lemma 7.1 can be repeated almost verbatim.

Proof of Lemma 7.2: The proof of the lemma has 11 steps. The first 10 steps prove the assertion of Lemma 7.2 regarding the case where $r < \frac{1}{8c} r_p$. The final step proves the assertion regarding the $N_{(\cdot)}(0) > \frac{1}{2}$ assumption for points near a given vertex.

Step 1: To start the proof of Lemma 7.2, note first that the depiction of the function N using the second bullet in (7.4) implies that it is an almost everywhere differentiable function on $(0, \frac{1}{1000}\pi]$, and that its derivative at its differentiable points has the form

$$(7.12) \quad \frac{dN}{dr} = -\frac{(1+2N)N}{r} + \frac{r}{AK^2} \int_{\partial B_r} |\nabla \psi|^2 + r z(r),$$

with $z(r)$ denoting a continuous function of r with norm bounded by $c_0(1+E)$. (This bound for $z(r)$ follows directly from definition of $K^2(r)$ as the average of $|\psi|^2$ on ∂B_r .)

Step 2: Use of a certain symmetric tensor on $S^3 - \Gamma$ (a section of $T^*S^3 \otimes T^*S^3$) is the key idea for analyzing the right-hand side of (7.12). (The introduction of this tensor and its application in very similar contexts is ultimately due to Almgren, but see also [7, 11, 12].) This tensor is denoted by T and it is defined via its components with respect to a chosen orthonormal frame for T^*S^3 . These are the functions $\{T_{ab}\}_{a,b=1,2,3}$ with any given T_{ab} defined below:

$$(7.13) \quad T_{ab} = \langle \nabla_a \psi, \nabla_b \psi \rangle + \langle \nabla_b \psi, \nabla_a \psi \rangle - \delta_{ab} |\nabla \psi|^2.$$

Here, ∇_a is the directional derivative along the dual basis vector to the 1-form labeled by the subscript a . The key observation for what follows is that the covariant divergence of T has the schematic form below (repeated indices are summed over the index set $\{1, 2, 3\}$)

$$(7.14) \quad \nabla_a T_{ab} = \langle \psi, \mathcal{Z}_{ab} \nabla_a \psi \rangle + \langle \mathcal{Z}_{ab} \psi, \nabla_a \psi \rangle.$$

The notation here uses $\{\mathcal{Z}_{ab}\}_{a,b=1,2,3}$ to denote a collection of endomorphisms of $\mathbb{V}$ whose norms are bounded by $c_0(1+E)$.

Step 3: Use a Gaussian coordinate system based at p for the radius $\frac{1}{100}\pi$ ball centered at p to define a Euclidean coordinate chart for this ball. Let $\{x_1, x_2, x_3\}$ denote these Euclidean coordinates. The 1-forms

$$\left\{ \frac{1}{1+|x|^2} dx_a \right\}_{a=1,2,3}$$

are an orthonormal basis for T^*S^3 on this ball. Use this basis to define the function $x_b \nabla_a T_{ab}$ on the ball (as in (7.14), repeated indices are summed).

Were p in $S^3 - \Gamma$ and were r less than the distance from p to Γ , then integrating the function $x_b \nabla_a T_{ab}$ over the radius r ball centered at p and then integrating by parts leads to the identity

$$(7.15) \quad r \int_{\partial B_r} (|\nabla_r \psi|^2 - |\nabla^\partial \psi|^2) = - \int_{B_r} |\nabla \psi|^2 + \mathfrak{C}(r)$$

with the notation being as follows: This identity is using $\nabla_r \psi$ to denote the covariant derivative of ψ in the normal direction of ∂B_r and it is using $\nabla^\partial \psi$ to denote the covariant derivative of ψ in the directions tangent to ∂B_r . Meanwhile, $\mathfrak{C}$ in (7.15) denotes a function of r with the following norm bound:

$$(7.16) \quad |\mathfrak{C}(r)| \leq c_0 \left(r^2 \int_{B_r} |\nabla \psi|^2 + \int_{B_r} |\psi|^2 \right).$$

To be sure: The restriction on p and r in the derivation of (7.15) is invoked so that the integration by parts that leads to (7.15) can't get hung up on Γ .

The explanation for why (7.15) holds when the closure of B_r intersects Γ is given in subsequent steps; but (in some cases) with a bound for the corresponding version of $\mathfrak{C}$ that is not as strong as the bound in (7.16).

Step 4: Assuming (7.15) holds (whether or not $\mathfrak{C}$ obeys (7.16)), that identity can be used to rewrite (7.12) as done below:

$$(7.17) \quad \frac{dN}{dr} = -\frac{(1+2N)N}{r} + \frac{2r}{AK^2} \int_{\partial B_r} |\nabla_r \psi|^2 + \frac{1}{AK^2} \int_{B_r} |\nabla \psi|^2 - \frac{1}{AK^2} \mathfrak{C}(r) + r z(r).$$

To proceed from here: Write $\nabla_r \psi$ in (7.17) as $\nabla_r \psi - \frac{N}{r} \psi + \frac{N}{r} \psi$ and use the first bullet in (7.4) to write the identity above as

$$(7.18) \quad \frac{dN}{dr} = -\frac{N}{r} + \frac{2r}{AK^2} \int_{\partial B_r} \left| \nabla_r \psi - \frac{N}{r} \psi \right|^2 + \frac{1}{AK^2} \int_{B_r} |\nabla \psi|^2 - \frac{1}{AK^2} \mathfrak{C}(r) + r z(r).$$

With (7.18) in hand, use the depiction of N from the second bullet in (7.4) to write the terms in (7.18) with the B_r integral of $|\nabla \psi|^2$ in terms of N . Doing so leads to this inequality:

$$(7.19) \quad \frac{dN}{dr} = \frac{2r}{AK^2} \int_{\partial B_r} \left| \nabla_r \psi - \frac{N}{r} \psi \right|^2 - \frac{1}{AK^2} \mathfrak{C}(r) + r w(r)$$

with $w(\cdot)$ denoting a continuous function of r with a $c_0(1+E)$ norm bound.

Step 5: Suppose in this step that $\mathfrak{C}$ does obey the bound in (7.16). Noting that the function K is almost non-decreasing (this follows from (7.3) and from the first bullet in Lemma 7.1), it follows that

$$(7.20) \quad \int_{B_r} |\psi|^2 \leq c_0 r^3 K^2(r).$$

Use this last bound for the $|\psi|^2$ integral on the right-hand side of (7.16), and use the second bullet in (7.4) to write the $|\nabla \psi|^2$ integral on the right-hand side of (7.16) in terms of N . Doing both of those and then using the resulting bound for $\mathfrak{C}$ in (7.19) leads directly to the inequality for the derivative of N that is asserted by Lemma 7.2.

With the preceding understood: As noted in Step 3, the inequality in (7.16) holds when p is in $S^3 - \Gamma$ and when $r \leq \frac{1}{2c} r_p$. The next step explains why this is also the case when p is from the interior of an edge of Γ . Step 7

explains why this is the case when p is a vertex in Γ . The last step in the proof (Step 11) explains why (7.16) holds for the case under consideration in the final assertion of Lemma 7.2.

Step 6: To see about the cases when B_r intersects Γ (which is always the case if the point p is in Γ), fix $\varepsilon > 0$ but very small (the $\varepsilon \rightarrow 0$ limit will be taken eventually). Fix a smooth non-increasing function

$$\chi : \mathbb{R} \longrightarrow [0, 1]$$

that is equal to 1 on $(-\infty, \frac{1}{4}]$ and equal to 0 on $[\frac{3}{4}, \infty)$. For the purposes of this step, let

$$(7.21) \quad \alpha_\varepsilon := \chi \left(1 - \frac{1}{\varepsilon} \text{dist}(\cdot, \Gamma) \right).$$

To be sure, this function is equal to 1 where the distance to Γ is greater than ε ; and it is equal to zero where the distance to Γ is less than $\frac{1}{4}\varepsilon$. Also fix $\delta > 100\varepsilon$ but very small (the $\delta \rightarrow 0$ limit will be taken also), let $\mathcal{V}$ denote the set of vertices of the graph Γ , and

$$(7.22) \quad \beta_\delta := \chi \left(1 - \frac{1}{\delta} \text{dist}(\cdot, \mathcal{V}) \right),$$

where

$$(7.23) \quad \text{dist}(\cdot, \mathcal{V}) := \min_{v \in \mathcal{V}} \text{dist}(\cdot, v).$$

To be sure, this function is equal to 1 where the distance to each vertex of Γ is greater than δ and equal to 0 where the distance to some vertex is less than $\frac{1}{4}\delta$.

Now return to the Step 3 and instead of integrating $x_b \nabla_a T_{ab}$ over B_r , integrate the function $\alpha_\varepsilon \beta_\delta x_b \nabla_a T_{ab}$ over B_r . Then integrate by parts. (The function α_ε is used so that the support of the integrand has distance greater than $\frac{1}{4}\varepsilon$ from Γ ; and the function β_δ is used to further shrink the support so that its distance to any vertex of Γ is greater than $\frac{1}{4}\delta$). The result of this integration by parts is an ε -dependent version of (7.15) where $\mathfrak{C}(\cdot)$ can be written as a sum of three terms,

$$(7.24) \quad \mathfrak{C}(\cdot) = \mathfrak{C}_{1\varepsilon, \delta}(\cdot) + \mathfrak{C}_{2\varepsilon, \delta}(\cdot) + \mathfrak{C}_{3\varepsilon, \delta}(\cdot).$$

with the distinction between these terms as follows: The term $\mathfrak{C}_{1\varepsilon, \delta}$ denotes a function of r that is bounded by what is written on the right-hand side of (7.16); it is short-hand for an integral over B_r of terms with no derivatives on α_ε or β_δ . Meanwhile, $\mathfrak{C}_{2\varepsilon, \delta}$ signifies the function of r that contains the integration by parts terms that do have derivatives of α_ε but none of β_δ . And, $\mathfrak{C}_{3\varepsilon, \delta}$ contains the integration by parts terms that have derivatives of β_δ but none of α_ε . The terms $\mathfrak{C}_{2\varepsilon, \delta}$ and $\mathfrak{C}_{3\varepsilon, \delta}$ are depicted below in (7.25) with the notation as follows: The first bullet's depiction of $\mathfrak{C}_{2\varepsilon, \delta}$ uses ∂_Γ for the function $\text{dist}(\cdot, \Gamma)$, and it uses α'_ε to denote the function on S^3 that is obtained by evaluating the derivative of χ at the value of the function

$(1 - \frac{1}{\varepsilon} \text{dist}(\cdot, \Gamma))$. The second bullet has $\mathfrak{d}_{\mathcal{V}}$ denoting the function $\text{dist}(\cdot, \mathcal{V})$ and it defines β'_δ analogously to α'_ε .

$$\begin{aligned} \bullet \mathfrak{C}_{2\varepsilon, \delta}(r) &= \frac{1}{\varepsilon} \int_{\mathbf{B}_r} \beta_\delta \alpha'_\varepsilon (\nabla_a \mathfrak{d}_\Gamma) x_b T_{ab}. \\ \bullet \mathfrak{C}_{3\varepsilon, \delta}(r) &= \frac{1}{\delta} \int_{\mathbf{B}_r} \alpha_\varepsilon \beta'_\delta (\nabla_a \mathfrak{d}_{\mathcal{V}}) x_b T_{ab}. \end{aligned} \quad (7.25)$$

The key concern with regard to $\mathfrak{C}_{2\varepsilon, \delta}$ and $\mathfrak{C}_{3\varepsilon, \delta}$ is whether they have respective limits as ε and δ are taken to zero, and if so, whether those limits are zero. Because the $\mathfrak{C}_{3\varepsilon, \delta}$ analysis is relatively straightforward, it is considered first below.

Step 7: This step explains why the contribution to $\mathfrak{C}_{3\varepsilon, \delta}$ has limit zero as $\delta \rightarrow 0$. To see that this is so, note first that the integrand in the second bullet of (7.25) is supported in the union of the radius δ balls centered at the vertices of Γ . With that understood, let q denote a given vertex and let $\mathbf{B}'_\delta$ denote the radius δ ball centered at q . The contribution from $\mathbf{B}'_\delta$ is at most

$$(7.26) \quad c_0 r_p \int_{\mathbf{B}'_\delta} \frac{1}{\text{dist}(\cdot, q)} |\nabla \psi|^2.$$

The inequality depicted by the fourth bullet in (6.3) implies that (7.26) is no larger than $c_0 r_p$, and then it follows from this last fact (by the dominated convergence theorem) that the $\delta \rightarrow 0$ limit of (7.26) is zero. (Note that ε plays no role in this analysis.)

Step 8: The analysis of the integral in (7.25) for $\mathfrak{C}_{2\varepsilon, \delta}$ will exploit the fact that the integrand is supported where the distance to the vertices of Γ are greater than $\frac{1}{4}\delta$ but where the distance to some edge of Γ is less than ε . In particular, if ε is much smaller than δ , then contribution to the integral for $\mathfrak{C}_{2\varepsilon, \delta}$ in (7.25) comes from points that are much closer to some given edge of Γ than they are to any vertex.

The subsequent analysis of the contribution of a given edge from Γ to the integral for $\mathfrak{C}_{2\varepsilon, \delta}$ in (7.25) depends whether the point p is on that edge or not. Suppose henceforth in this step that p is not on a given edge that intersects $\mathbf{B}_r$. (Note that if $\mathbf{B}_r$ intersects that edge, then $r \geq c_0^{-1} r_p$.) Since the length of $\mathbf{B}_r$'s intersection with that edge is at most $c_0 r$, the part of that edge where β_δ is non-zero can be covered by $c_0 r \varepsilon^{-1}$ balls of radius ε centered at points on this edge. Let q now denote the center point of one of those balls and let $\mathbf{B}'_\varepsilon$ denote the radius ε ball centered at q . The contribution to $\mathfrak{C}_{2\varepsilon, \delta}$ in (7.25) from the ball $\mathbf{B}'_\varepsilon$ is at most

$$(7.27) \quad c_0 r \frac{1}{\varepsilon} \int_{\mathbf{B}'_\varepsilon} |\nabla \psi|^2.$$

To see how big this can be, multiply both sides of (6.5) by

$$\chi\left(\frac{1}{\varepsilon} \text{dist}(\cdot, q) - 1\right),$$

which is a function with compact support in $B'_{2\varepsilon}$ that is equal to 1 on B'_ε . Then integrate the resulting identity and after doing that, integrate by parts twice to remove derivatives from $|\psi|^2$ and put them on the cut-off function $\chi(\frac{1}{\varepsilon} \text{dist}(\cdot, q) - 1)$. The result of doing that leads to the bound below when ε is very small:

$$(7.28) \quad \int_{B'_\varepsilon} |\nabla \psi|^2 \leq c_0 \frac{1}{\varepsilon^2} \int_{B'_{2\varepsilon}} |\psi|^2.$$

Crucially, Lemma 6.1 can be brought to bear if $\varepsilon < \frac{1}{4c} r_q$; hence the right-hand side of (7.28) in this case is bounded by $c_0 \varepsilon$. Note that if $\varepsilon < \frac{1}{1000c} \delta$, then ε will be less than $\frac{1}{4c} r_q$ if the distance of q to the nearest vertex of Γ is greater than δ .

Supposing henceforth that ε is less than $\frac{1}{1000c} \delta$, then this $c\varepsilon$ bound for (7.27) implies a $c_0 r$ bound for the contribution to the integral for $\mathfrak{C}_{2\varepsilon, \delta}$ in (7.25) from an edge of Γ that does not contain the point p . Note in particular that this bound is independent of both ε and δ .

Step 9: This step considers the contribution to the integral for $\mathfrak{C}_{2\varepsilon, \delta}$ in (7.25) from an edge that contains p , but with the additional assumption that p is an interior point of that edge. It is also assumed in this step that r is less than $\frac{1}{8c} r_p$ so as to eventually bring Lemma 6.2 to bear. The exploitation of Lemma 6.2 uses the fact that if ε is much less than $\frac{1}{8c} r_p$, then on the support of α'_ε inside the ball of radius $\frac{1}{8c} r_p$ centered at p , the vector x_b is almost tangent to the edge in question. This means in particular that

$$(7.29) \quad |x_b \nabla_b \mathfrak{d}_\Gamma| < c_0 \varepsilon \quad \text{and} \quad |x_b - |x| \nu_b| < c_0 \varepsilon |x|,$$

where $\{\nu_b\}_{b=1,2,3}$ denotes the components of the vector field $\frac{\partial}{\partial t}$ that appears in Lemma 6.2.

Let $N_{\varepsilon, \delta}$ denote the part of the radius ε tubular neighborhood of the edge in question, where the distance to Γ is greater than $\frac{1}{4} \varepsilon$ and where the distance to the vertex set $\mathcal{V}$ of Γ is greater than $\frac{1}{4} \delta$. With $N_{\varepsilon, \delta}$ understood, it then follows as a consequence of the preceding bounds that the contribution to the integral for $\mathfrak{C}_{2\varepsilon, \delta}$ from the edge containing p is at most

$$(7.30) \quad c_0 r \frac{1}{\varepsilon} \left(\int_{N_{\varepsilon, \delta}} |\nabla \psi|^2 \right)^{1/2} \left(\int_{N_{\varepsilon, \delta}} |\nabla_t \psi|^2 \right)^{1/2} + c_0 r \int_{N_{\varepsilon, \delta}} |\nabla \psi|^2.$$

The rightmost integral in (7.30) has limit 0 as $\varepsilon \rightarrow 0$ because $|\nabla \psi|$ is in the Hilbert space $\mathbb{H}$. With regards to the rightmost integral: Lemma 6.2 says that $\nabla_t \psi$ is in $\mathbb{H}$, and bounds its $\mathbb{H}$ -norm by $c_0 \frac{1}{\sqrt{r_p}}$. It follows from this fact and (4.2) that

$$(7.31) \quad \int_{N_{\varepsilon, \delta}} |\nabla_t \psi|^2 \leq c_0 \varepsilon^2 \frac{1}{r_p}.$$

Use of the latter in the leftmost term of (7.30) bounds the latter by

$$(7.32) \quad c_0 \frac{r}{\sqrt{r_p}} \left(\int_{N_{\varepsilon,\delta}} |\nabla \psi|^2 \right)^{1/2},$$

which has limit zero as $\varepsilon \rightarrow 0$. To summarize what was just proved: If p is in the interior of an edge of Γ and if $r < \frac{1}{8c} r_p$, then the term $\mathfrak{C}_{2\varepsilon,\delta}$ has limit zero as $\varepsilon \rightarrow 0$.

Step 10: Now consider the case when p is a vertex of Γ with $r < \frac{1}{8c} r_p$. In this case, the integral for $\mathfrak{C}_{2\varepsilon,\delta}$ in (7.25) is the sum of contributions from the part of the radius ε tubular neighborhood of p 's incident edges that have a distance between $\frac{1}{4}\delta$ and r from p . If ε is much smaller than δ , then those tubular neighborhoods are disjoint. In addition, the bounds in (7.29) hold on each such neighborhood. As a consequence, the analysis in Step 9 can be repeated with almost no changes to see that the $\varepsilon \rightarrow 0$ limit of $\mathfrak{C}_{2\varepsilon,\delta}$ is zero.

Step 11: Suppose in this step that q is a given vertex of Γ and that there exists $\rho > 0$ such that $N_{(\cdot)}(0) > \frac{1}{2}$ at each point in Γ with distance ρ or less from q . The plan for this case is to show that if $p \in S^3$ has distance less than $\frac{1}{2}\rho$ from q , then the inequality for the derivative of N holds on $(0, c_0^{-1}\rho]$. This will be followed by a demonstration that the integral in (7.25) for $\mathfrak{C}_{2\varepsilon,\delta}(r)$ has limit zero as $\varepsilon \rightarrow 0$ for fixed δ when $r \leq c_0^{-1}\rho$.

To see about this limit, the first observation is the conclusions of Lemma 7.1 can be assumed and, in particular, the conclusions of the third bullet since those conclusions only require that the differential inequality from Lemma 7.2 holds at points in the interior of an edge and for $r < \frac{1}{8c} r_{(\cdot)}$. This being the case, it then follows that for any give value of δ , there exists $\mu > 0$ such that if p is a point with distance between $\frac{1}{8}\delta$ and ρ from the vertex q , then $N_p(0) \geq \frac{1+\mu}{2}$. It then follows from this that using (7.3) that the integral on the right-hand side of (7.28) is bounded by $c_0 \varepsilon^{2+\mu} r_p^{-(1+\mu)}$. Since r_p is greater than $c_0^{-1}\delta$ by assumption, this in turn is bounded by $c_0 \varepsilon^{2+\mu} \delta^{-(1+\mu)}$.

Since the domain of integration for (7.25)'s integral defining $\mathfrak{C}_{2\varepsilon,\delta}$ can be covered by $c_0 \varepsilon^{-1}$ balls of radius ε centered at points on edges of Γ , the bound $|\mathfrak{C}_{2\varepsilon,\delta}| \leq c_0 \varepsilon^{1+\mu} \delta^{-(1+\mu)}$ follows directly from the preceding analysis. The $\varepsilon \rightarrow 0$ limit of $\mathfrak{C}_{2\varepsilon,\delta}$ is therefore zero for any fixed choice of δ as required.

8. Proofs for Lemmas 4.2 and 4.4

This section uses parts of Lemmas 7.1–7.3 to prove Lemmas 4.2 and 4.4.

Proof of Lemma 4.2: The proof explains why $|\psi|$ is Hölder continuous near Γ with Hölder exponent that is no less than the minimum of $\frac{1}{2}$ and the versions of the number $N(0)$ that are defined by the vertices of Γ . (The second bullet of Lemma 7.1 says, in part, that this minimum is positive.) This explanation has 5 steps, most riffing on discourses in [23]. Even so, the details are written. With regards to notation: Given a positive number

r and a point in S^3 , the notation is such that B_r again denotes the radius r ball centered at that point.

Step 1: Suppose that $d > 0$ and that q is a point in S^3 with $\text{dist}(q, \Gamma) = d$. As explained directly, the equation in (6.4) on the radius d ball centered at q when written as

$$(8.1) \quad \nabla^\dagger \nabla \psi = (\mathbf{E} - \mathcal{R})\psi$$

implies the inequalities below for the integral of $|\nabla \psi|^2$ on the radius $\frac{31}{32}d$ ball centered at q , and for the integral of $|\nabla^{\otimes 2} \psi|^2$ on the radius $\frac{30}{32}d$ ball centered at q :

$$\begin{aligned} \bullet \int_{B_{31d/32}} |\nabla \psi|^2 &\leq c_0(1 + \mathbf{E})^2 \frac{1}{d^2} \int_{B_d} |\psi|^2, \\ \bullet \int_{B_{30d/32}} |\nabla^{\otimes 2} \psi|^2 &\leq c_0(1 + \mathbf{E})^2 \frac{1}{d^4} \int_{B_d} |\psi|^2. \end{aligned} \quad (8.2)$$

To obtain the first bullet of (8.2): Construct a bump function (to be denoted by ς_d) with values in $[0, 1]$ that is equal to 0 near the boundary of the radius d ball centered at q and equal to 1 on the concentric radius $\frac{31}{32}d$ ball with respective $c_0 \frac{1}{d}$ and $c_0 \frac{1}{d^2}$ bounds on its differential and Hessian. Write the identity in (6.4) as $\nabla^\dagger \nabla \psi = (\mathbf{E} - \mathcal{R})\psi$, take the inner product of both sides with $\varsigma_d \psi$ and then integrate over B_d . Having done that, then integrate by parts on the left-hand side. To obtain the second bullet of (8.2): Construct a second version of ς_d that equals 0 on the complement of that concentric, radius $\frac{31}{32}d$ ball and equals 1 on the concentric, radius $\frac{30}{32}d$ ball. Take the square of the norm of both sides of the identity $\nabla^\dagger \nabla \psi = (\mathbf{E} - \mathcal{R})\psi$, multiply the result by this new version of ς_d , and then integrate over B_d . Two applications of integration by parts on the left-hand side leads to the second bullet of (8.2), given what is said by (8.2)'s first bullet.

Step 2: What follows directly is an application of (8.2): Suppose that x and y are distinct points in S^3 with $\text{dist}(x, y) < \frac{1}{100}\pi$. Given x and y , let d denote a number that obeys the bound $\frac{30}{32}d > \frac{1}{2} \text{dist}(x, y)$ which is imposed so that both x and y are in the radius $\frac{30}{32}d$ ball centered at the midpoint of the geodesic arc between them. Let ψ denote for the moment a section of $\mathbb{V} \otimes \mathcal{I}$ over the radius d ball centered at the midpoint of the short geodesic between x and y with $|\nabla^{\otimes 2} \psi|^2$ having finite integral on the concentric radius $\frac{30}{32}d$ ball. Then the Hölder norm bound below holds

$$(8.3) \quad ||\psi(x)| - |\psi(y)|| \leq c_0 \text{dist}(x, y)^{1/2} \left(\int_{B_{30d/32}} |\nabla^{\otimes 2} \psi|^2 + \frac{1}{d^4} \int_{B_{30d/32}} |\psi|^2 \right)^{1/2},$$

this being a consequence of dimension 3 Sobolev inequalities. (See e.g. [10] for the latter).

Suppose now that ψ obeys (6.4) and that the radius d ball centered at the midpoint between x and y is disjoint from Γ . According to (8.2), the function $|\nabla^{\otimes 2} \psi|^2$ does have finite integral over $B_{30d/32}$; and then (8.3) with

(8.2) lead to the next bound:

$$(8.4) \quad ||\psi(x)| - |\psi(y)|| \leq c_0 \operatorname{dist}(x, y)^{1/2} \frac{1}{d^2} \left(\int_{B_{30d/32}} |\psi|^2 \right)^{1/2}.$$

Hold on to the preceding inequality for the moment.

Step 3: Supposing that p is a given point in $S^3 - \Gamma$, let p' denote a closest point to p in Γ , the distance from p to p' being the number r_p . Suppose in this step and the next step that r_p (which is $\operatorname{dist}(p, p')$) is less than $\frac{1}{8c} r_{p'}$ with c defined by (6.1). Thus the point p is in one of the cones depicted in Fig. 1 and p' is from the interior of one of Γ 's edges.

Letting γ denote the short geodesic between p and p' , set x_1 to denote the point on γ half-way between p and p' . Thus, $\operatorname{dist}(p, x_1) = \frac{1}{2} \operatorname{dist}(p, p')$. Set $d = \frac{1}{2} r_p$. Of particular note is that $\frac{30}{32}d > \frac{1}{2} \operatorname{dist}(p, x_1)$ and that the radius d ball centered at the midpoint between x_1 and p is disjoint from Γ . As a consequence, (8.4) can be invoked with $x = x_1$ and $y = p$.

To say more about that version of (8.4), note also that the radius d ball centered on the midpoint between x_1 and p is contained in the radius $\frac{3}{2}r_p = 3d$ ball centered at p' . Noting that the latter ball is inside the radius $\frac{1}{4c} r_{p'}$ ball centered at p' , Lemma 7.3 can be brought to bear to bound the integral that appears in this version of (8.4). Bring that lemma to bear results in a bound of the form

$$(8.5) \quad ||\psi(x_1)| - |\psi(p)|| \leq c_0 z_{p'} \operatorname{dist}(x_1, p)^{N(0)} \leq c_0 z_{p'} \frac{1}{\sqrt{2}} \operatorname{dist}(p, p')^{N(0)}.$$

with $z_{p'}$ coming from the p' version of Lemma 7.3 and with $N(0)$ coming from the p' version of Lemma 7.1. (Lemma 7.1 says in part that this version of $N(0)$ is no smaller than $\frac{1}{2}$).

With regards to $z_{p'}$: Let q denote the closest vertex to p' so as to define Lemma 7.3's number $\bar{N} = \max(0, N(0) - N_q(0))$ and then bound $z_{p'}$ by $c_0/r_{p'}^{\bar{N}}$. Granted the latter bound, and granted that $\operatorname{dist}(p, p') \leq c_0 r_{p'}$, the bound in (8.5) leads directly to the Hölder bound:

$$(8.6) \quad ||\psi(x_1)| - |\psi(p)|| \leq \frac{c_0}{\sqrt{2}} \operatorname{dist}(p, p')^\mu \leq c_0 \operatorname{dist}(x_1, p)^\mu,$$

with μ denoting the lesser of the numbers $\frac{1}{2}$ and the versions of $N(0)$ that are defined by the vertices of Γ . For use below: Introduce c_* to denote the version of the number c_0 that appears in (8.6).

To continue, let x_2 denote the point along γ half way between x_1 and p' . The analysis that lead to (8.6) can be redone with x_1 replaced by x_2 and p replaced by x_1 to see that

$$(8.7) \quad ||\psi(x_2)| - |\psi(x_1)|| \leq c_* \operatorname{dist}(x_2, x_1)^\mu = \frac{c_*}{2^{2\mu}} \operatorname{dist}(p, p')^\mu.$$

Continue sequentially in this same vein with x_n for $n \geq 3$ denoting the point on γ with distance to p' being $2^{-n} \text{dist}(p, p')$ to see that

$$(8.8) \quad ||\psi(x_n)| - |\psi(x_{n-1})|| \leq c_* \text{dist}(x_n, x_{n-1})^\mu = \frac{c_*}{2^{n\mu}} \text{dist}(p, p')^\mu.$$

Since the sequence $\{x_n\}_{n=1,2,\dots}$ converges to p' , adding the corresponding versions of (8.8) leads directly to the Hölder bound

$$(8.9) \quad ||\psi(p)| - |\psi(p')|| \leq c_* \text{dist}(p, p')^\mu.$$

Note that (8.9) implies that the function $|\psi|$ is Hölder continuous with exponent μ along the normal direction at any point from the interior of any edge of Γ . (Of course, this also follows directly from Lemma 6.1.)

Step 4: Lemma 6.1 says that $|\psi|$ extends across the interiors of the edges of Γ to be zero on Γ . This step explains why $|\psi|$ is Hölder continuous with exponent μ on a neighborhood of any point in the interior of an edge of Γ . By way of background for this explanation: Suppose that q is an interior point on an edge of Γ , and that points x and y are in the radius $\frac{1}{128c} r_q$ ball centered at q . Let q_x and q_y denote the respective closest points to x and y in Γ . These will be in the interior of the same edge as q and with their corresponding numbers r_{q_x} and r_{q_y} being greater than $\frac{1}{4} r_q$. It follows as a consequence of what is said by (8.9) that

$$(8.10) \quad ||\psi(x)| - |\psi(q_x)|| \leq c_0 \text{dist}(x, q_x)^\mu \quad \text{and} \quad ||\psi(y)| - |\psi(q_y)|| \leq c_0 \text{dist}(y, q_y)^\mu.$$

By way of more background: Let ρ denote the smaller of $\text{dist}(x, q_x)$ and $\text{dist}(y, q_y)$. If $\text{dist}(x, y) < \frac{1}{1000} \rho$, then (8.3) can be brought to bear with $d = \frac{1}{100} \rho$; and that inequality with the $r = \frac{1}{10} \rho$ version of Lemma 7.3 leads to the Hölder bound

$$(8.11) \quad ||\psi(x)| - |\psi(y)|| \leq c_0 \text{dist}(x, y)^\mu.$$

The inequalities in (8.10) and (8.11) can now be used to obtain a uniform, exponent μ Hölder bound for $|\psi|$ near the given point p . Indeed, this comes directly from (8.11) in the event that $\text{dist}(x, y) < \frac{1}{1000} \rho$ where ρ again denotes the minimum of $\text{dist}(x, q_x)$ and $\text{dist}(y, q_y)$. (Implicit here and in what follows in this step is that both x and y are in the radius $\frac{1}{128c} r_q$ ball centered at some given point q in the interior of one of Γ 's edges.) In the event that $\text{dist}(x, y) \geq \frac{1}{1000} \rho$, the desired Hölder bound is obtained from the inequality

$$(8.12) \quad ||\psi(x)| - |\psi(y)|| \leq |\psi(x)| + |\psi(y)|.$$

To see how to the desired Hölder bound follows from (8.12), let q_x and q_y again denote the respective closest points to x and y in Γ . Since both $|\psi(q_x)|$ and $|\psi(q_y)|$ are zero, the inequality in (8.12) leads via (8.10) to the one below:

$$(8.13) \quad ||\psi(x)| - |\psi(y)|| \leq c_0 \text{dist}(x, q_x)^\mu + c_0 \text{dist}(y, q_y)^\mu.$$

To proceed from (8.13) to the desired Hölder bound, assume (without loss of generality) that

$$(8.14) \quad \text{dist}(x, q_x) \leq \text{dist}(y, q_y).$$

Noting that $\text{dist}(y, q_y) \leq \text{dist}(y, q_x)$ because q_y is the closest point to y in Γ , and noting also that

$$(8.15) \quad \text{dist}(y, q_x) \leq \text{dist}(x, y) + \text{dist}(x, q_x)$$

which is less than $1001 \text{dist}(x, y)$, it then follows from (8.13) that

$$(8.16) \quad ||\psi(x)| - |\psi(y)|| \leq 2\sqrt{1001} c_0 \text{dist}(x, y)^\mu.$$

The bound above is the promised bound for when $\text{dist}(x, y)$ is not less than $\frac{1}{1000}$ times the minimum of $\text{dist}(x, \Gamma)$ and $\text{dist}(y, \Gamma)$.

Step 5: Suppose in this step that p is from $S^3 - \Gamma$ and that $r_p = \text{dist}(p, \Gamma)$ is not less than any version of $\frac{1}{8c} r_q$ when q is a closest point to p in Γ . In this event, let q henceforth denote a chosen, closest vertex to p . If r_p is greater than this vertex q 's version of $\frac{1}{8c} r_q$, then the distance from p to Γ is no less than c_0^{-1} which is to say that p is quite far from Γ in which case p is of no concern for purposes at hand. Therefore, suppose in what follows that $r_p < \frac{1}{8c} r_q$ with q denoting the chosen closest vertex from Γ . This assumption and the initial assumption in this step imply that p lies in the ball depicted in Fig. 1, but that it is not inside the cones centered on the edges from the vertex q . An important point to keep in mind for what follows is that $\text{dist}(p, \Gamma) \geq c_0^{-1} \text{dist}(p, q)$, this being a consequence of the positive lower bound for the solid angle cross sections of the cones that are depicted in Fig. 1, and it is closer to the vertex center of that sphere than it is to the edge that is tangent to the disk. Let c_* denote the larger of this last version of c_0 and the number c from (6.1).

To proceed in this case, let γ denote the short geodesic from q to p and let x_1 denote the point on γ that is halfway between that vertex q and p . Let

$$(8.17) \quad d = \frac{1}{1000 c_*} \text{dist}(p, q).$$

Fix equally spaced points along γ with distance $d/4$ apart, starting with p such that the last one has distance $d/4$ or less from x_1 . Let N denote the number of such points noting in particular that $N \leq 1000c_*$. Denote these points in order along γ moving from p to x_1 by $y_1, \dots, y_N$ and then set y_{N+1} to be x_1 . Now, for each integer k starting with $k = 1$ and ending with N , bring (8.4) to bear sequentially with $x = y_k$ and $y = y_{k+1}$ keeping in mind that Lemma 7.3 can be invoked with p the given vertex because of (8.17). The result is the bound below:

$$(8.18) \quad ||\psi(y_k)| - |\psi(y_{k+1})|| \leq c_0 \text{dist}(p, q)^\mu.$$

This implies in turn that

$$(8.19) \quad ||\psi(x_1)| - |\psi(p)|| \leq c_0 \operatorname{dist}(p, q)^\mu.$$

Let x_2 denote the point half way along γ between x_1 and q . Repeat the preceding operation with x_1 replaced by x_2 and with p replaced by x_1 . Having done that, repeat again with x_2 replaced by the point half way along γ between x_2 and q , and with x_1 replaced by x_2 . Continuing sequentially in this way leads to a sequence $\{p, x_1, x_2, \dots\}$ that converges to q and such that the following holds for each positive integer:

$$(8.20) \quad ||\psi(x_n)| - |\psi(x_{n+1})|| \leq c_0 \frac{1}{2^{\mu n}} \operatorname{dist}(p, q)^\mu.$$

Of course, these imply that

$$(8.21) \quad ||\psi(p)| - |\psi(q)|| \leq c_0 \operatorname{dist}(p, q)^\mu.$$

Coupled with the results of the previous step, this last inequality implies that $|\psi(q)| = 0$. Thus, $|\psi|$ vanishes at the vertices Γ also. The arguments from the previous step can also be repeated with minor changes to see that $|\psi|$ is uniformly Hölder continuous with exponent μ on a neighborhood of each vertex.

Proof of Lemma 4.4: The proof has four steps.

Step 1: It follows from the assumptions and from Lemmas 7.1 and 7.2 that any $q \in \Gamma$ version of the number $N(0)$ is no smaller than 1. Indeed, if $N(r)$ were less than $1 - \varepsilon$ for any given sufficiently small but positive r , then Lemma 7.1's second bullet would imply that

$$(8.22) \quad N(r) < 1 - c_0^{-1} \varepsilon$$

for all smaller values of r . Then, by virtue of the definition of N via (7.3), this would run afoul of the assumption that the function $r \rightarrow \frac{1}{r} K(r)$ is bounded as $r \rightarrow 0$. Thus, since $N(\cdot)$ is (almost) a non-decreasing function of r (see Lemma 7.2), it follows (again from (7.3)) that the inequalities

$$(8.23) \quad \int_{\partial B_r} |\psi|^2 \leq c_0 r^4 \quad \text{and} \quad \int_{B_r} |\psi|^2 \leq c_0 r^5$$

hold for all $r \in (0, c_0^{-1}]$. Since the rightmost bound is uniform with respect to the choice of the point in Γ (see Lemma 7.3), that bound implies in turn the following bound:

$$(8.24) \quad \int_{\operatorname{dist}(\cdot, \Gamma) \leq r} |\psi|^2 \leq c_0 r^4.$$

(The exponent of r drops from 5 to 4 because Γ has non-zero, finite 1-dimensional Hausdorff measure.) Hold onto this inequality for the moment.

Step 2: Let σ denote a favorite, non-decreasing $[0, 1]$ -valued function on $\mathbb{R}$ that is equal to 0 on $(-\infty, \frac{1}{4}]$ and equal to 1 on $[\frac{3}{4}, \infty)$. Given a positive

number r , use this function to define a function on S^3 (denoted in what follows by σ_r) by the rule

$$(8.25) \quad \sigma_r(\cdot) = \sigma\left(\frac{1}{r} \operatorname{dist}(\cdot, \Gamma)\right).$$

Write the identity in (6.4) as $\nabla^\dagger \nabla \psi = (\mathbf{E} - \mathcal{R})\psi$, multiply each side by the function σ_r , and note that it implies in turn the identity

$$(8.26) \quad \nabla^\dagger(\sigma_r \nabla \psi) = (\mathbf{E} - \mathcal{R}) \sigma_r \psi - \nabla \sigma_r \cdot \nabla \psi.$$

Of particular import is that $\sigma_r \nabla \psi$ has compact support on $S^3 - \Gamma$ (as does each term on the right-hand side of (8.26)).

Take the square of the norm of both sides of the identity in (8.26) and then integrate the result over S^3 . An integration by parts with an appeal to the top bullet in (6.3) leads to the following inequality:

$$(8.27) \quad \int_{S^3} |\nabla(\sigma_r \nabla \psi)|^2 \leq c_0(1 + \mathbf{E})^2 + c_0 \frac{1}{r^2} \int_{\frac{r}{4} < \operatorname{dist}(\cdot, \Gamma) \leq r} |\nabla \psi|^2.$$

Step 3: To see about the integral on the right-hand side of (8.27), do as before and write (6.4) as $\nabla^\dagger \nabla \psi = (\mathbf{E} - \mathcal{R})\psi$, then take the inner product of both sides of that identity with ψ , then multiply the result by $\sigma_{r/8}(1 - \sigma_{8r})$. Having done all of that, integrate the result over S^3 . An integration by parts then results in the inequality below:

$$(8.28) \quad \int_{\frac{r}{4} < \operatorname{dist}(\cdot, \Gamma) \leq r} |\nabla \psi|^2 \leq c_0 \frac{1}{r^2} \int_{\frac{r}{8} < \operatorname{dist}(\cdot, \Gamma) \leq 2r} |\psi|^2.$$

Step 4: Use (8.28) in (8.27) to bound the latter's right-hand side integral, and then invoke (8.24) to obtain the r -independent bound below for the $\mathbb{L}$ -norm of $\nabla(\sigma_r \nabla \psi)$:

$$(8.29) \quad \int |\nabla(\sigma_r \nabla \psi)|^2 \leq c_0(1 + \mathbf{E})^2.$$

It now follows by taking r to zero in (8.29) that $\nabla(\nabla \psi)$, the covariant derivative of $\nabla \psi$ is in the Hilbert space $\mathbb{L}$. This implies in turn that $\nabla \psi$ is in $\mathbb{H}$.

9. Fourier modes of ψ near an edge of Γ

Fix the vector bundle $\mathbb{V}$ and its connection; and then let ψ denote a $(\nabla^\dagger \nabla + \mathcal{R})$ -eigensection on $S^3 - \Gamma$ with $\mathbb{L}$ -norm equal to 1. There is a standard way to use the properties of the frequency function $\mathbf{N}$ to ‘zoom’ in on the behavior of ψ near any given point in Γ . This zoom in process for a point in the interior of a given edge of Γ is described in this section.

To do this zoom-in, let p denote a point on the interior of an edge. Choose a Gaussian coordinate chart centered at p to identify the ball of radius $\frac{1}{8c} r_p$ centered at p with the same radius ball in $\mathbb{R}^3$. To be more specific: This chart should be chosen so that the edge containing p corresponds to the

part of the x_3 -axis where x_3 is between $-\frac{1}{8c}r_p$ and $\frac{1}{8c}r_p$. The restriction of the line bundle $\mathcal{I}$ to the radius $\frac{1}{8c}r_p$ ball centered at p is identified by these coordinates with the real line bundle that is defined on the complement in this ball of the x_3 axis with monodromy -1 on linking circles with the x_3 -axis.

With $\mathbb{V}$ now viewed as a vector bundle with connection over the radius $\frac{1}{8c}r_p$ ball centered at the origin in $\mathbb{R}^3$, pick an orthonormal frame for $\mathbb{V}$ at the origin and use parallel transport out along the rays in $\mathbb{R}^3$ from the origin to write $\mathbb{V}$ as a product bundle. The fiber of that product vector bundle is denoted by V in what follows. Doing that identifies a section of $\mathbb{V}$ over that ball with a map from the $|x| < \frac{1}{8c}r_p$ ball to V . Likewise, a section of $\mathbb{V} \otimes \mathcal{I}$ over the complement of the x_3 -axis in the $\frac{1}{8c}r_p$ ball centered at p will be viewed as a section of the vector bundle $\mathcal{I} \times V$.

Supposing that $\lambda \in (0, 1]$, define the rescaling map $\phi_\lambda : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by the rule whereby $x = (x_1, x_2, x_3) \mapsto \lambda x$. Parallel transport on radial geodesics from the points on the $|x| = 1$ sphere in $\mathbb{R}^3$ identifies the line bundle $\mathcal{I}$ with its pull-back via ϕ_λ . This identification is taken for granted henceforth. If ψ denotes a section of the bundle $\mathcal{I} \times V$ on the complement of the x_3 -axis in the $|x| \leq \frac{1}{8c}r_p$ ball, then $\phi_\lambda^* \psi$ (which is the composition of first ϕ_λ and then ψ) defines a section of $\mathcal{I} \times V$ on the complement of the x_3 -axis in the $|x| \leq \frac{1}{8c}\lambda^{-1}r_p$ ball.

Given $\lambda \in (0, \frac{1}{16c}r_p]$, define the section ψ_λ on this same domain by the rule whereby

$$(9.1) \quad x \mapsto \psi_\lambda(x) = \frac{1}{K(\lambda)} \psi(\lambda x).$$

The square of what is denoted by $K(\lambda)$ in (9.1) is the $r = \lambda$ version of what is depicted on the right-hand side of (7.1). The division above by $K(\lambda)$ is done so that the average of $|\psi_\lambda|^2$ over the $|x| = 1$ sphere in $\mathbb{R}^3$ equals 1.

Define a ψ_λ version of the function K with domain $(0, \frac{1}{8c}\lambda^{-1}r_p)$ to be the positive square root of the function depicted below:

$$(9.2) \quad r \mapsto K_\lambda(r)^2 = \frac{1}{A(r)} \int_{\partial B_r} |\psi_\lambda|^2,$$

where B_r denotes here the $|x| \leq r$ ball in $\mathbb{R}^3$ and ∂B_r denotes the $|x| = r$ boundary sphere. Meanwhile, define the function N_λ on the same domain as K_λ by writing

$$(9.3) \quad \frac{dK_\lambda}{dr} = \frac{N_\lambda(r)}{r} K_\lambda(r).$$

Of particular note is the fact that K_λ and ψ 's version of the function K (defined using the point p), and N_λ and ψ 's version of the function N (also defined using p) are related as follows:

$$(9.4) \quad K_\lambda(r) = K(\lambda)^{-1} K(\lambda r) \quad \text{and} \quad N_\lambda(r) = N(\lambda r).$$

Granted (9.4), then Lemma 7.1's second bullet can be brought to bear to justify the following observation:

Having fixed $R \geq 2$ and $\varepsilon > 0$, there exists a positive number $\lambda_{R,\varepsilon}$ such that $|\mathbf{N}_\lambda(\cdot) - \mathbf{N}(0)| \leq \varepsilon$ on $[0, R]$ when $\lambda \leq \lambda_{R,\varepsilon}$.

(9.5)

This is to say that as λ limits to zero, the values of $\mathbf{N}_\lambda$ converge to $\mathbf{N}(0)$ uniformly on any *a priori* bounded subset of $[0, \infty)$. This convergence implies in turn (via (9.3)) the following convergence assertion for $\mathbf{K}_\lambda$:

Having fixed $R \geq 2$ and $\varepsilon > 0$, there exists a positive number, $\lambda'_{R,\varepsilon}$ such that $\left| \frac{1}{r^{N(0)}} \mathbf{K}_\lambda(r) - 1 \right| \leq \varepsilon$ for all $r \in \left[\frac{1}{R}, R \right]$ when $\lambda \leq \lambda'_{R,\varepsilon}$.

(9.6)

The convergence of $\mathbf{N}_\lambda$ and $\mathbf{K}_\lambda$ have implications for the integral of $|\nabla \psi_\lambda|^2$, the following is the first (it comes from (9.4)–(9.6) and (7.4)):

Having fixed $R \in (1, \infty)$ and $\varepsilon > 0$, there exists a positive number, $\lambda''_{R,\varepsilon}$ such that when $\lambda \leq \lambda''_{R,\varepsilon}$ and $r \in \left[\frac{1}{R}, R \right]$, then

$$\left| \frac{1}{r^{2N(0)+1}} \int_{\mathbf{B}_r} |\nabla \psi_\lambda|^2 - N(0) \right| \leq \varepsilon.$$

(9.7)

In this last equation and subsequently, the connection that defines the co-variant derivative ∇ is the product connection on the product V bundle.

The inequality in (8.2) leads (via coordinate rescaling) to *a priori* bounds on the second derivatives of ψ_λ in an annulus around the x_3 -axis. To write this precisely, fix $R > 2$ and let A_R denote the subset of $\mathbb{R}^3$ where the Euclidean coordinates (x_1, x_2, x_3) are such that

$$(9.8) \quad \frac{1}{R} < (x_1^2 + x_2^2)^{1/2} < R \quad \text{and} \quad |x_3| < R.$$

The bound below holds for any given $R \geq 2$ when λ is sufficiently small (the number c_R below depends on R but not on λ):

$$(9.9) \quad \int_{A_R} |\nabla^{\otimes 2} \psi_\lambda|^2 \leq c_R R^{2N(0)}.$$

Moreover, given $R > 2$, there are also λ -independent bounds for sufficiently small λ on the integral over A_R of the square of the norms of the derivatives of ψ_λ to any given order:

$$(9.10) \quad \int_{A_R} |\nabla^{\otimes k} \psi_\lambda|^2 \leq c_{R,k} R^{2N(0)}.$$

where $c_{R,k}$ depends on R and k , but not λ . Indeed, these follow using standard elliptic regularity tools, given that ψ_λ where $|x| \leq \frac{1}{8c} \lambda^{-1} r_p$ obeys the rescaled version of (6.4)'s eigenvalue equation, this being an equation that has the schematic form depicted below:

$$(9.11) \quad \nabla^\dagger \nabla \psi_\lambda + \lambda(a_\lambda \cdot \nabla \psi_\lambda) + \lambda^2 \mathcal{T}_\lambda \psi_\lambda - \lambda^2 \mathbf{E} \psi_\lambda = 0.$$

Here, the norms of a_λ and $\mathcal{T}_\lambda$ obey $|a_\lambda| \leq c_0 \lambda |x|$ and $|\mathcal{T}_\lambda| \leq c_0$; and the norms of their derivatives obey $|\nabla^{\otimes k} a_\lambda| \leq c_{*,k} \lambda^{k+1} (1 + |x|)$ and $|\nabla^{\otimes k} \mathcal{T}_\lambda| \leq c_{*,k} \lambda^k$ with $c_{*,k}$ signifying a constant that depends on k but not on λ .

Another critically important bound follows from Lemma 7.2, which is the one below:

Having fixed $R > 2$ and $\varepsilon > 0$, there exists a positive number, $\lambda''_{R,\varepsilon}$ such

$$\text{that when } \lambda \leq \lambda''_{R,\varepsilon} \text{ then } \int_{\mathbf{B}_R - \mathbf{B}_{1/R}} \left| \nabla_r \psi_\lambda - \frac{\mathbf{N}(0)}{r} \psi_\lambda \right|^2 \leq \varepsilon. \quad (9.12)$$

And, there is also the critically important bound which follows from Lemma 6.2 regarding the derivative of ψ_λ in the x_3 direction:

If $R > 2$, and assuming that λ is sufficiently small given R , then

$$(9.13) \quad \int_{\mathbf{B}_R} |\nabla_3 \psi_\lambda|^2 \leq c_0 \lambda.$$

By way of an explanation for this last bound: The rescaling that defines ψ_λ from ψ identifies the integral in (9.13) with an integral of $\frac{1}{\lambda} |\nabla_\nu \psi|^2$ over the radius λR ball in S^3 centered at the original point p up to an $\mathcal{O}(\lambda R^2 \kappa^2(\lambda R))$ error which is due to the choice of the connection that defines the covariant derivative in (9.13). This identification results initially in the bound below (the notation has ν denoting the vector field in Lemma 6.2):

$$(9.14) \quad \int_{\mathbf{B}_R} |\nabla_3 \psi_\lambda|^2 \leq c_0 \frac{1}{\lambda} \int_{\mathbf{B}_{\lambda R}} |\nabla_\nu \psi|^2 + c_1 \lambda R^2 \kappa^2(\lambda R).$$

Meanwhile, Lemma 6.2 gives an *a priori* bound for the integral of the square of the covariant derivative of $\nabla_\nu \psi$ over any small radius ball centered at p ; and that bound with the inequality in (6.11) which holds now when ψ is replaced by $\nabla_\nu \psi$ implies (via a dimension 3 Sobolev inequality for $|\nabla_\nu \psi|$) that the integral of $|\nabla_\nu \psi|^6$ over $\mathbf{B}_{\lambda R}$ has an R and λ independent upper bound. Since

$$(9.15) \quad \int_{\mathbf{B}_{\lambda R}} |\nabla_\nu \psi|^2 \leq c_0 (\lambda^3 R^3)^{2/3} \left(\int_{\mathbf{B}_{\lambda R}} |\nabla_\nu \psi|^6 \right)^{1/3},$$

that uniform upper bound on the integral of $|\nabla_\nu \psi|^6$ over $\mathbf{B}_{\lambda R}$ with (9.15) leads directly from (9.14) to the asserted bound in (9.13).

The preceding *a priori* bounds imply convergence of subsequences of the various ψ_λ 's as $\lambda \rightarrow 0$. The lemma below makes a precise statement and describes the limit.

LEMMA 9.1. Suppose that $\{\lambda_j\}_{j=1,\dots}$ is a decreasing sequence of λ -values with limit zero. There is a subsequence whose corresponding sequence of ψ_λ 's converges to a non-zero, section of $\mathcal{I} \times V$ in the C^∞ topology on compact subsets of the complement in $\mathbb{R}^3$ of the x_3 -axis. This subsequence also converges in both the $\mathbb{L}$ and $\mathbb{H}$ topologies for the space of sections of $\mathcal{I} \times V$ over any ball in $\mathbb{R}^3$. Moreover, the limit (denoted by $\psi_\dagger$) is harmonic in the sense that $\nabla^\dagger \nabla \psi_\dagger = 0$. It is also invariant with respect to translations in the x_3

direction; and it is equivariant under coordinate rescalings in the sense that $\phi_\lambda^* \psi_\dagger = \lambda^{N(0)} \psi_\dagger$.

Proof of Lemma 9.1. Fix $R > 1$. It follows from (9.7) that the set $\{\psi_\lambda\}_{\lambda \in (0,1)}$ has an *a priori* $\mathbb{H}$ -norm bound on the $|x| \leq R$ ball in $\mathbb{R}^3$. This implies that any sequence from this set with λ converging to zero along the sequence has a subsequence (hence denoted by $\{\psi_i\}_{i=1,2,\dots}$) that converges weakly in the $\mathbb{H}$ topology and strongly in the $\mathbb{L}$ -topology (the Rellich Lemma) on this ball. By taking diagonal subsequences for $R = 2, 3, \dots$, no generality is lost by requiring that $\{\psi_i\}_{i=1,2,\dots}$ converge weakly in the $\mathbb{H}$ -topology on any $R > 1$ version of the $|x| < R$ ball and strongly in the $\mathbb{L}$ -topology on any such ball. What with (9.6), the Rellich lemma also implies that the limit section $\psi_\dagger$ is non-zero and, more to the point, that

$$(9.16) \quad \int_{B_r} |\psi_\dagger|^2 = 4\pi r^{2N(0)+3}.$$

The convergence in (9.6) with (9.11) also implies that $\{\psi_i\}_{i=1,2,\dots}$ converges to $\psi_\dagger$ on any $|x| < R$ ball in the strong $\mathbb{H}$ -topology.

In addition: The bounds supplied by (9.9) and (9.10) imply that $\{\psi_i\}_{i=1,2,\dots}$ converges in the C^∞ topology on any $R > 1$ version of the set A_R . That fact coupled with (9.11) implies that $\psi_\dagger$ is a smooth section of $\mathcal{I} \times V$ on the complement of the x_3 -axis in $\mathbb{R}^3$ and that it obeys $\nabla^\dagger \nabla \psi_\dagger = 0$. Meanwhile (9.12) implies that $\nabla_r \psi_\dagger = \frac{N(0)}{r} \psi_\dagger$ and (9.13) implies that $\nabla_3 \psi_\dagger = 0$.

Let $\psi_\dagger$ denote a $\lambda \rightarrow 0$ limit from Lemma 9.1 of a sequence from the set $\{\psi_\lambda\}_{\lambda \in (0,1]}$. The fact that $\psi_\dagger$ is $\nabla^\dagger \nabla$ harmonic and that it is equivariant under coordinate rescalings and that it is annihilated by ∇_3 determines any given $\mathcal{I} \times V$ component of $\psi_\dagger$. To say more, let ψ_* denote a component of $\psi_\dagger$, thus a section of $\mathcal{I}$ over the complement in $\mathbb{R}^3$ of the x_3 -axis. Write $\mathbb{R}^3$ as $\mathbb{C} \times \mathbb{R}$ with the $\mathbb{R}$ factor corresponding to the x_3 coordinate and with the $\mathbb{C}$ factor having the complex coordinate $z = x_1 + ix_2$. Let u_0 denote a unit length, constant section of $\mathcal{I}$ over the complement of the x_3 -axis and the negative real axis (where $x_1 < 0$). Then ψ_* must have the form

$$(9.17) \quad \psi_* = (\alpha z^{N(0)} + \bar{\alpha} \bar{z}^{N(0)}) u_0,$$

with α denoting a complex number. This restriction on ψ_* implies in turn (since ψ_* is a section of $\mathcal{I}$ and u_0 is constant) that $N(0)$ must come from the set $\{\frac{1}{2}, \frac{3}{2}, \dots\}$.

10. Proof of Theorem 1.2

To set some background for the subsequent arguments: Having chosen one of the polytopes listed in Theorem 1.2, center that polytope in $\mathbb{R}^4$ so that its symmetry group is a finite subgroup of the group $\text{SO}(4)$ with the latter acting linearly on $\mathbb{R}^4$. Let G denote that subgroup. Now let Γ denote the graph in S^3 that is given by the union of the radial projection to S^3 of the set of edges and vertices of that polytope. The group G acts on $S^3 - \Gamma$

and, according to Proposition 11.12 in Section 11, it lifts in the manner described in Section 5 to an action on the line bundle $\mathcal{I}$. Moreover, these lifts are such that any given edge of Γ is fixed by a $\mathbb{Z}/m\mathbb{Z}$ subgroup with m being an odd integer and this $\mathbb{Z}/m\mathbb{Z}$ subgroup induces a linear action on the normal 2-plane to any interior point of that edge (this is a $2\pi/m$ rotation of that normal plane).

The three paragraphs that follow directly are here to give some indication of how the analysis of the preceding sections and the group theory are used in Theorem 1.2's proof. To set notation, let ψ in these paragraphs denote an eigensection of the operator $\nabla^\dagger \nabla + R$ on $S^3 - \Gamma$ from the Hilbert space $\mathbb{H}$. The preceding analysis established, via Eq. (4.2), that the number $N(0)$ as defined by ψ and for any given point on Γ is non-zero. (By way of a reminder, $N(0)$ is the $r \rightarrow 0$ limit of Almgren's frequency function $N(r)$ for that point. See (7.4).) Basically, the monodromy -1 property for the components of ψ on the linking circles to the edges of Γ force $N(0)$ to be positive. This is Item b) and c) of the second bullet in Lemma 7.1. No symmetry property is used for that; it is a consequence of the -1 monodromy which forces the components of ψ to have zeros on any linking circle. (For a vertex of Γ , this is saying in effect that the minimum eigenvalue for the 2-sphere Laplacian on sections of $\mathcal{I}$ over the complement of Γ on any small radius, linking 2-sphere to any vertex is necessarily positive.) The fact that $N(0)$ is positive implies that $|\psi|$ is Hölder continuous across Γ ; this being the content of Lemma 4.2.

Now we also need to know that the $\nabla\psi$ versions of $N(0)$ are also positive for any point on Γ . In this regard, $\nabla\psi$ also obeys an eigenvalue equation $\nabla^* \nabla(\nabla\psi) + R' \nabla\psi = E \nabla\psi$, with R' being a curvature endomorphism (the Ricci curvature if ψ is a section of $\mathcal{I}$). We can apply the arguments for Lemmas 4.2 and 7.1 to $\nabla\psi$ if we know that $|\nabla(\nabla\psi)|^2$ is integrable across Γ . Lemma 4.4 asserts in effect that this will occur if ψ 's version of $N(0)$ is greater than 1 at each point of Γ . For points in the interior of an edge of Γ , it is sufficient to know that $N(0)$ is greater than 1 because that implies that the integral of $|\psi|^2$ on the radius- r ball centered at that point is bounded by some multiple of r^5 when r is sufficiently small; and as explained in the proof of Lemma 4.4 starting on page 45, that is sufficient to conclude that $|\nabla(\nabla\psi)|^2$ is integrable across Γ . This is where the symmetries of the polytopes are used in the event that ψ is an invariant section of $\mathcal{I}$; they are used to conclude that actually $N(0)$ is at least $3/2$ for points in the interior of edges. The fact that $N(0)$ is greater than 1 for interior points to edges with the assertion of the fourth bullet in Lemma 7.1 implies that $N(0)$ for any vertex is not smaller than the smallest value of $N(0)$ on incident edges (thus $3/2$). In this regard: the $N(0) \geq 1$ condition for edges is used in the proof of the fourth bullet in Lemma 7.1 to justify an integration by parts; if $N(0) < 1$, then the relevant integration by parts can get hung up with divergent terms along incident edges.

As is evident from what transpires next in Section 10, the argument in the case of self-dual, $\mathbb{Z}/2$ harmonic 2-forms does require much more by

way of analysis regarding the linking 2-spheres to the vertices. This detailed analysis is the content of the seven parts of Section 10. The reason for the difficulty in this case is that a homogeneous, self-dual harmonic 2-form on $\mathbb{R}^4$ is defined by a coclosed, $\mathcal{I}$ -valued eigensection on $S^3 - \Gamma$ for the operator $*d$ (as opposed to the elliptic $*d + *d*$). Getting the coclosed condition and, simultaneously, the Hölder continuity of the derivatives requires far more analysis of the behavior of invariant, $\mathcal{I}$ -valued 1-form near the edges and near vertices. That analysis explicitly leverages the symmetry group invariance via the subgroups that fix given edges and vertices.

Theorem 1.2 makes claims regarding homogeneous, $\mathbb{Z}/2$ harmonic 1-forms, self-dual spinors and self-dual 2-forms. The argument for the claims regarding $\mathbb{Z}/2$ harmonic 1-forms and self-dual spinors is considered first. To start that argument, invoke Lemma 5.1 to obtain a non-trivial, G -invariant Laplace eigensection from $\mathcal{I}$'s version on S^3 of the Hilbert space $\mathbb{H}$. This section is denoted by ψ and its eigenvalue is denoted by E . This is to say that ψ obeys the equation $-\Delta^\perp \psi = E\psi$. To continue, let q denote an interior point to an edge of Γ and use this point to ‘zoom in’ on the behavior of ψ near q as was done in Section 9. Doing that results in a version of ψ_* that has the form depicted in (9.17). Keeping in mind that the edge is fixed by a $\mathbb{Z}/m\mathbb{Z}$ subgroup of G , what is said in Section 5 (see the paragraph prior to Lemma 5.1) implies that the version of $N(0)$ that appears in (9.17) must come from the set $\{\frac{m}{2}, \frac{3m}{2}, \dots\}$. Since m is at least 3, it then follows from the manner of convergence dictated in Section 9 that the conditions for bringing Lemma 4.4 to bear are met by this section ψ , which implies in particular that $d\psi$ is in the Hilbert space $\mathbb{H}$ and as such, it is a $\nabla^\dagger \nabla + \text{Ric}$ eigensection from the $T^*S^3 \otimes \mathcal{I}$ version of the Hilbert space $\mathbb{H}$. (Ric here denotes the Ricci curvature when viewed as an endomorphism of T^*S^3 .) In particular, $|d\psi|$ extends across Γ as a Hölder continuous function that vanishes on Γ .

Now let r denote the radial coordinate on $\mathbb{R}^4$ and let π denote the radial projection map from $\mathbb{R}^4 \setminus \{0\}$ to S^3 . Let $\kappa \equiv (1 + E)^{1/2} - 1$. Then $\phi = r^\kappa \pi^* \psi$ is a homogeneous, $\mathbb{Z}/2$ harmonic function on $\mathbb{R}^4$ whose corresponding version of the set Z is the closure of $\pi^{-1}\Gamma$ and whose corresponding real line bundle is the π -pull-back of the bundle $\mathcal{I}$ on S^3 . (That real line bundle is subsequently denoted by $\mathcal{I}$ also.) The exterior derivative of ϕ (thus $d\phi$) is the promised homogeneous, harmonic, $\mathcal{I}$ -valued 1-form on $\mathbb{R}^4$ if E is greater than 3 (because then $|d\phi|$ limits to zero as $r \rightarrow 0$). With regards to this lower bound for E : Lemmas 5.1 and 4.1 guarantee an infinite set of E in $(3, \infty)$.

To obtain a homogeneous, $\mathcal{I}$ -valued, harmonic self-dual spinor, let η denote a given, non-zero, constant anti-self-dual spinor on $\mathbb{R}^4$. Supposing that $\mathfrak{D}$ denotes the standard Dirac operator on $\mathbb{R}^4$ mapping anti-self-dual spinors to self-dual spinors (and vice-versa), then $\mathfrak{D}(\phi\eta)$ is a homogeneous, harmonic, $\mathcal{I}$ self-dual spinor on $\mathbb{R}^4$ if E is greater than 3.

The construction of Theorem 1.2's homogeneous $\mathbb{Z}/2\mathbb{Z}$ harmonic, self-dual 2-forms requires more work. The following seven parts of this section exhibit that construction.

Part 1: The plan for what follows is to construct a version of what is denoted by $\nu^\perp$ in (2.9) for certain positive values of κ . This $\nu^\perp$ will be a section of the $T^*S^3 \otimes \mathcal{I}$ version of the Hilbert space $\mathbb{H}$ whose norm extends across Γ as a Hölder continuous function that vanishes on Γ . Granted $\nu^\perp$, then the self-dual, $\mathcal{I}$ valued 2-form that is depicted in (2.7) is the desired, homogeneous, $\mathcal{I}$ -valued harmonic self-dual 2-form for Theorem 1.2.

The construction of $\nu^\perp$ starts by invoking Lemma 5.2 to obtain G -invariant, $(\nabla^\dagger \nabla + \text{Ric})$ -eigensections in the $T^*S^3 \otimes \mathcal{I}$ version of the Hilbert space $\mathbb{H}$. Let ψ denote such an eigensection and let $\mathbf{E}$ denote its eigenvalue. Let d denote for the purposes of this proof the exterior derivative on S^3 and let $*$ denote the Hodge star on S^3 . Granted this notation, set $\phi = *d*\psi$, this being a section of $\mathcal{I}$ which is a priori only in the Hilbert space $\mathbb{L}$. But note in this regard that if each edge of Γ is fixed by a $\mathbb{Z}/m\mathbb{Z}$ subgroup of G with m odd and at least 5, then (by virtue of (5.9) and what is said in Section 9), the assumptions required by Lemma 4.4 will be met and as a consequence, ϕ will come from $\mathcal{I}$'s version of $\mathbb{H}$.)

Anyway, with regards to ϕ , this obeys the equation $-\Delta^\perp \phi = \mathbf{E}\phi$ because the eigenvalue equation $(\nabla^\dagger \nabla + \text{Ric})\psi = \mathbf{E}\psi$ can be written as

$$(10.1) \quad -(*d(*d\psi) + d(*d*\psi)) = \mathbf{E}\psi,$$

and because $\Delta^\perp \phi = *d(*d\phi)$. Note also that ϕ is a G -invariant section of $\mathcal{I}$.

If it is the case that ϕ is from $\mathbb{H}$ (which is guaranteed if $m > 3$), then $d\phi$ is also in $\mathbb{H}$. This is a consequence of Lemma 4.4 whose requirements are met by ϕ because ϕ is a G -invariant, $(-\Delta^\perp)$ -eigensection from $\mathbb{H}$. Indeed, being a $-\Delta^\perp$ -eigensection from $\mathbb{H}$ implies that ϕ has its versions of (9.17), and being G -invariant implies that the values of $\mathbf{N}(0)$ that appear there are no smaller than $\frac{3}{2}$ (see Section 5).

Supposing that ϕ is in $\mathbb{H}$ and thus $d\phi$ too, it then follows that $y = \psi - \frac{1}{\mathbf{E}}d\phi$ is a coclosed section of $\mathbb{H}$ that obeys

$$(10.2) \quad *d(*dy) = \mathbf{E}y.$$

Since y is coclosed, it is a G -invariant, $(\nabla^\dagger \nabla + \text{Ric})$ -eigensection from the $T^*S^3 \otimes \mathcal{I}$ version of $\mathbb{H}$. As explained in subsequent parts of this proof, $*dy$ is also from $\mathbb{H}$ (this follows directly from (9.17) and (5.9) if $m > 3$). Granted that $*dy$ is from $\mathbb{H}$, then so is $y + \frac{1}{\sqrt{\mathbf{E}}} *dy$. This is the desired section $\nu^\perp$ for use in (2.9). (The number κ is $\sqrt{\mathbf{E}}$.)

With regards to the situation when $m = 3$: In that case, the top bullet of (5.9) allows for the case $k = 1$ which implies that ψ 's version of $\mathbf{N}(0)$ can be equal to $\frac{1}{2}$ which is too small to guarantee *a priori* that ϕ (which is $*d*\psi$) is from $\mathbb{H}$. Even so, the manner of convergence to the rescaling limits of ψ near the interior points of the edges of Γ as described in Section 9 with what is said by (5.11) suggests that ϕ is indeed in $\mathbb{H}$ notwithstanding the fact that $\nabla\psi$ is not in $\mathbb{H}$. This same manner of convergence to rescaling limits with (5.11) also suggests that $*dy$ is in $\mathbb{H}$ notwithstanding the fact that $\nabla\psi$ is not.

Part 2: This part of the proof and Parts 3 and 4 explain why ϕ is indeed in $\mathbb{H}$. To start the explanation, bring Lemma 5.1 to bear so as to obtain a G -invariant section from $\mathcal{I}$'s version of the Hilbert space $\mathbb{H}$ (to be denoted by $\mathbf{v}$) that obeys the equation $-\Delta^\perp \mathbf{v} = \mathbf{E}\phi$. Let $\varsigma = \phi - \mathbf{v}$. This section ς is a $\Delta^\perp$ -harmonic section of $\mathcal{I}$ from the Hilbert space $\mathbb{L}$ since $\Delta^\perp \phi$ and $\Delta^\perp \mathbf{v}$ are both $\mathbf{E}\phi$. A key point now is that $\varsigma \equiv 0$ if ς is in $\mathbb{H}$; and if $\varsigma \equiv 0$, then $\phi = \mathbf{v}$ and so ϕ is from $\mathbb{H}$ as claimed.

The question of whether or not ς is from $\mathbb{H}$ concerns its behavior near Γ since it is smooth on $S^3 - \Gamma$. To see about this, the rest of this part of the proof and all of Part 3 focus attention on some given point in the interior of an edge of Γ . This point is denoted subsequently by p . Rotate the 3-sphere in $\mathbb{R}^4$ if needed so that the edge containing p is an arc in the $|x| = 1$ part of the $x_3 = x_4$ plane in $\mathbb{R}^4$. Let $z = (x_3 + ix_4)$ and then write $x_1 + ix_2$ as $(1 - |z|^2)^{1/2} e^{it}$. The $\mathbb{R}/2\pi\mathbb{Z}$ valued coordinate t is chosen so that p is the $t = 0$ point on the $|z| = 0$ great circle. It proves convenient to write z in polar form as $z = \rho e^{i\theta}$ with $\rho > 0$ and θ being $\mathbb{R}/2\pi\mathbb{Z}$ valued. The metric on a ball of radius less than π in S^3 when written using the coordinates ρ , θ and t has the form

$$(10.3) \quad \frac{1}{(1 - \rho^2)} d\rho^2 + \rho^2 d\theta^2 + (1 - \rho^2) dt^2.$$

As noted in the proof of Lemma 6.2, the constant translations of t are isometries. (These identify the normal disks to the given edge of Γ . The constant rotations of these normal disks (thus $\theta \rightarrow \theta + \mu$ with μ constant) are also isometries and they commute with those given by constant translations of t . These isometries will be exploited momentarily.)

Fix a constant, unit-length section of $\mathcal{I}$ in the part of the $\rho < \frac{1}{1024c} r_p$ and $|t| < \frac{1}{1024c} r_p$ polydisc where $\theta \in (-\pi, \pi)$. Use that section to depict ς on that part of the polydisc as an $\mathbb{R}$ -valued function. Fourier series can then be used to write ς in this polydisc as a sum,

$$(10.4) \quad \varsigma = \sum a_{k,n}(\rho) e^{i(k+\frac{1}{2})\theta} e^{i1024\pi c n t / r_p},$$

this sum being a double sum over the values of k from $\mathbb{Z}$ and the values of n from $\mathbb{Z}$. With regards to this Fourier decomposition: Keep in mind that ς is smooth on the complement of the $\rho = 0$ locus and it is also in $\mathbb{L}$ (which is to say that $|\varsigma|^2$ has finite integral on S^3). It follows as a consequence that the sum in (10.4) converges pointwise assuming ρ is positive and both ρ and $|t|$ are less than $\frac{1}{1024c} r_p$.

To say more about the functions $a_{k,n}(\cdot)$: Note first that $\bar{a}_{k,n} = a_{-k-1,-n}$ since ς is $\mathbb{R}$ -valued. A second point of note: Because ς is harmonic, any given $a_{k,n}$ obeys the second order ordinary differential equation

$$(10.5) \quad -\frac{1}{\rho} \frac{d}{d\rho} \left(\rho(1 - \rho^2) \frac{d}{d\rho} a_{k,n} \right) + \frac{(k + \frac{1}{2})^2}{\rho^2} a_{k,n} + \frac{(\pi n / 2r)^2}{(1 - \rho^2)} a_{k,n} = 0,$$

with r denoting here $\frac{1}{1024c} r_p$. A key point regarding the preceding equation (which can be derived using standard theorems about ODEs) is that any solution must have the form

$$(10.6) \quad a_{k,n}(x) = (\alpha_{k,n} + \rho^2 \mathfrak{f}_{k,n}) \rho^{k+\frac{1}{2}} + (\beta_{k,n} + \rho^2 \mathfrak{h}_{k,n}) \rho^{-(k+\frac{1}{2})},$$

with $\alpha_{k,n}$ and $\beta_{k,n}$ being constant, and with $|\mathfrak{f}_{k,n}|$ and $|\mathfrak{h}_{k,n}|$ being bounded functions of ρ . (This is because $\rho = 0$ is the only singular point of that equation.) Since ς is in $\mathbb{L}$, the depiction of $a_{k,n}$ in (10.6) implies in particular (by taking $\rho \ll 1$) that $\alpha_{k,n} = 0$ for $k \leq -2$ and $\beta_{k,n} = 0$ for $k \geq 1$. But there is more: Since ς is G invariant, $\alpha_{k,n}$ and $\beta_{k,n}$ must be zero unless $2k+1 = 0 \bmod m$. These conditions require (in particular) that $\alpha_{-1,n} = \alpha_{0,n} = 0$ and that $\beta_{0,n} = \beta_{-1,n} = 0$.

Because of the preceding constraints, the norm of $a_{k,n}$ obeys

$$(10.7) \quad |a_{k,n}| \leq z_{k,n} \rho^{3/2},$$

with $z_{k,n}$ being constant. Here is more to keep in mind about the set $\{a_{k,n}\}$: The fact that $|\varsigma|^2$ has finite integral implies that the integral below is finite

$$(10.8) \quad \sum \int_0^{\frac{r_p}{1024c}} |a_{k,n}|^2 \rho d\rho \leq c_0(1 + \mathbf{E}),$$

where the sum is over the same set as in (10.4).

Part 3: Given $\varepsilon > 0$, let σ_ε denote a smooth function of the coordinate ρ that is equal to 1 where $\rho < \varepsilon$ and equal to 0 where $\rho > 2\varepsilon$; and also with $|d\sigma_\varepsilon| \leq c_0 \varepsilon^{-1}$. The constructions that follow use σ_p to denote the $\varepsilon = \frac{1}{2048c} r_p$ version of σ_ε . Multiply both sides of (10.5) by $(1 - \sigma_\varepsilon)\sigma_p \bar{a}_{k,n}$ and then integrate the result. Since the integrand has compact support on $(0, \frac{1}{1024c} r_p)$, an integration by parts can be done (and (10.7) can be invoked) to obtain the following inequality:

$$(10.9) \quad \int_0^{\frac{r_p}{2048c}} \left(\left| \frac{d}{d\rho} a_{k,n} \right|^2 + \frac{(k + \frac{1}{2})^2}{\sin^2 \rho} |a_{k,n}|^2 + \frac{(\frac{\pi n}{2r})^2}{\cos^2 \rho} |a_{k,n}|^2 \right) \rho d\rho \\ \leq c_0 |z_{k,n}|^2 \varepsilon^3 + \frac{c_0}{r_p^2} \int_0^{\frac{r_p}{1024c}} |a_{k,n}|^2 \rho d\rho.$$

With (10.9) in hand: Having fixed $\delta > 0$ and a large integer N , it follows from (10.9) and (10.8) that there exists $\varepsilon_{\delta,N} > 0$ such that if $\varepsilon < \varepsilon_{\delta,N}$, then

$$(10.10) \quad \sum^N \int_0^{\frac{r_p}{2048c}} \sigma_\varepsilon \left(\left| \frac{d}{d\rho} a_{k,n} \right|^2 + \frac{(k + \frac{1}{2})^2}{\sin^2 \rho} |a_{k,n}|^2 + \frac{(\frac{\pi n}{2r})^2}{\cos^2 \rho} |a_{k,n}|^2 \right) \rho d\rho \\ \leq \delta + \frac{c_0}{r_p^2} (1 + \mathbf{E}),$$

where the symbol $\sum^N$ indicates there is a sum over pairs (k, n) such that $|k| < N$ and $|n| < N$. Another important point with regards to (10.9):

Having fixed $\delta > 0$ then that inequality also implies that there exists positive $\varepsilon'_{k,N}$ such that if $\varepsilon < \varepsilon'_{k,N}$, and $N' > N$, then

$$(10.11) \quad \sum_{N,N'} \int_0^{r_p/2048c} \sigma_\varepsilon \left(\left| \frac{d}{d\rho} a_{k,n} \right|^2 + \frac{(k + \frac{1}{2})^2}{\sin^2 \rho} |a_{k,n}|^2 + \frac{(\pi n/2r)^2}{\cos^2 \rho} |a_{k,n}|^2 \right) \rho d\rho < \delta,$$

where the symbol $\sum_{N,N'}$ indicates that the sum is over all pairs (k, n) with $|k|$ and $|n|$ between N and N' . The observations in (10.10) and (10.11) imply this: Set $r = \frac{1}{1024c} r_p$ and set χ_p to denote for now the function $\chi(2^{10} \text{dist}(p, \cdot)/r - 1)$. Then, the sequence

$$(10.12) \quad \left\{ \chi_p \sum_{N=1}^N a_{k,n}(\rho) e^{i(k+\frac{1}{2})\theta} e^{i\pi 1024c n t/r_p} : N = 1, 2, \dots \right\}$$

is a Cauchy sequence in the B_r version of the Hilbert space $\mathbb{H}_B$. This last conclusion implies in turn that $\chi_p \varsigma$ is in $\mathbb{H}$ since this sequence converges in the B_r version of $\mathbb{L}_B$ to $\chi_p \varsigma$.

Part 4: To summarize: Part 3 established that $|\nabla \varsigma|^2$ is integrable on the complement in S^3 of any open set containing the vertices of Γ . With this understood, the focus in this part of the proof is on the behavior of $\nabla \varsigma$ near some given vertex of Γ . Use p now to denote that vertex. Fix $\varepsilon \in (0, \frac{1}{2048c} r_p)$ but very small. Let χ_ε and χ_p denote the respective $r = \varepsilon$ and $r = \frac{1}{2048c} r_p$ versions of the function $\chi(\text{dist}(\cdot, p)/r - 1)$. Since $-\Delta^\perp \varsigma = 0$, it is also true that

$$(10.13) \quad -\frac{1}{2} \Delta^\perp |\varsigma|^2 + |\nabla \varsigma|^2 = 0.$$

Multiply both sides of this last identity by $(1 - \chi_\varepsilon)\chi_p$ and then integrate the result over the radius $\frac{1}{1024c} r_p$ ball centered at p . Two instances of integration by parts leads from that integral identity to the inequality below:

$$(10.14) \quad \int (1 - \chi_\varepsilon)\chi_p |\nabla \varsigma|^2 \leq \frac{c_0}{\varepsilon^2} \int (1 - \chi_{\varepsilon/16})\chi_{2\varepsilon} |\varsigma|^2 + c_{p,E},$$

where $c_{p,E}$ is a positive number that is determined by p and E . This last inequality leads directly to an *a priori* bound for the integral of $|\nabla \varsigma|^2$ on the radius $\frac{1}{4096c} r_p$ ball centered at p if, for every small but positive ε , there exists a $c_0 \varepsilon^2$ bounds for the integral of ς on radius 2ε ball centered at p .

Find such a bound: Let B_r denote for the moment the $r = \frac{1}{1024c} r_p$ radius ball in S^3 centered at p . Let v denote the valency of the vertex p . The edges of Γ that are incident to p intersect ∂B_r as a configuration of v distinct points; denote this set by

$$(10.15) \quad \mathfrak{p} := \Gamma \cap \partial B_r.$$

A chosen Gaussian coordinate system centered at p identifies ∂B_r with the radius r sphere centered at the origin in $\mathbb{R}^3$, and then the radial projection from the origin in $\mathbb{R}^3$ identifies that sphere with the radius 1 sphere in $\mathbb{R}^3$. That radius 1 sphere is denoted by S^2 . Under this identification, the image of $\mathfrak{p} \subset \partial B_r$ is a configuration of v points in S^2 , which we also denote by $\mathfrak{p}$. Radial pull-back from ∂B_r of the line bundle $\mathcal{I}$ defines a real line bundle on $S^2 - \mathfrak{p}$ which will be denoted by $\mathcal{I}$ also.

Borrowing from [27], let $\mathbb{H}_{\mathfrak{p}}$ denote the Hilbert space that is obtained by taking the completion of the space of compactly supported sections of the $S^2 - \mathfrak{p}$ version of $\mathcal{I}$ using the norm whose square is the functional

$$(10.16) \quad \mathfrak{f} \mapsto \int_{S^2} |d\mathfrak{f}|^2.$$

Define the Hilbert space $\mathbb{L}_{\mathfrak{p}}$ to be the completion of that same space of sections of $\mathcal{I}$ on $S^2 - \mathfrak{p}$ using the norm whose square sends a section to the S^2 integral of the square of its norm.

Let Δ_2 denote the Laplace operator on S^2 for the round metric but with it acting on sections of $\mathcal{I}$ over $S^2 - \mathfrak{p}$. The section $\mathfrak{f}$ of $\mathcal{I}$ on $S^2 - \mathfrak{p}$ is said to be an eigensection in $\mathbb{H}_{\mathfrak{p}}$ for $-\Delta_2$ when it obeys the equation $-\Delta_2 \mathfrak{f} = \lambda \mathfrak{f}$ with λ being a non-negative real number. (The number λ is $\mathfrak{f}$'s eigenvalue.) Proposition 1.1 from Taubes-Wu [27] makes an assertion to the effect that the set of eigenvalues of these eigensections from $\mathbb{H}_{\mathfrak{p}}$ is discrete, bounded away from zero and has no accumulation points. This same theorem also asserts that each such eigenvalue has finite multiplicity. Meanwhile, Proposition 1.1 [27] makes an assertion to the effect that $\mathbb{L}_{\mathfrak{p}}$ has an orthonormal basis of $(-\Delta_2)$ -eigensections from $\mathbb{H}_{\mathfrak{p}}$. Let $\{\mathfrak{f}_j\}_{j=1,2,\dots}$ denote this orthonormal basis, labeled using the convention that the eigenvalue of any given version of $\mathfrak{f}_j$ is no larger than that of $\mathfrak{f}_{j+1}$.

Returning to ς : Let $r = \frac{1}{1024c} r_p$ and let B_r denote again the radius r ball centered at p . Let ρ now denote the function on B_r giving the distance to p . The section ς can be written on B_r using that orthonormal basis of $(-\Delta_2)$ -eigensections from $\mathbb{H}_{\mathfrak{p}}$ as

$$(10.17) \quad \varsigma = \sum \mathfrak{a}_j(\rho) \mathfrak{f}_j$$

where the sum is over the indexing set $j \in \{1, 2, \dots\}$ for the $\mathbb{L}_{\mathfrak{p}}$ -orthonormal basis of $(-\Delta_2)$ -eigensections. The coefficient function set $\{\mathfrak{a}_j(\rho)\}_{j=1,2,\dots}$ obey the ordinary differential equation

$$(10.18) \quad -\frac{1}{\sin^2 \rho} \frac{d}{d\rho} \left(\sin^2 \rho \frac{d}{d\rho} \mathfrak{a}_j \right) + \lambda_j \frac{1}{\sin^2 \rho} \mathfrak{a}_j = 0.$$

Because $|\varsigma|^2$ is integrable on S^3 , the function set $\{\mathfrak{a}_j\}$ are constrained to obey the bound

$$(10.19) \quad \sum \int_0^r |\mathfrak{a}_j|^2 \sin^2 \rho \, d\rho \leq c'_{p,E}$$

with $c'_{p,E}$ denoting another positive number that depends only on p and E .

Fix a non-negative integer j and set $\mathfrak{a} \equiv \mathfrak{a}_j$ and $\lambda \equiv \lambda_j$. Then, introduce the numbers

$$(10.20) \quad \mu \equiv \frac{1}{2} \left(-1 + (1 + 4\lambda)^{1/2} \right) \quad \text{and} \quad \mu' \equiv \frac{1}{2} \left(1 + (1 + 4\lambda)^{1/2} \right).$$

Standard ordinary differential equation technology can be used to see that the function $\mathfrak{a}(\cdot)$ must have the form in (10.21):

$$(10.21) \quad \mathfrak{a}(\rho) = \alpha(1 + \rho^2 \mathfrak{s})\rho^\mu + \beta(1 + \rho^2 \mathfrak{t})\rho^{-\mu'},$$

where α and β are j -dependent real numbers, and $\mathfrak{s}$ and $\mathfrak{t}$ are j -dependent functions of ρ obeying $|\mathfrak{s}| + |\mathfrak{t}| \leq c_j$ with c_j being a positive real number.

Granted (10.21), it then follows that the contribution to the small ε integral in (10.14) from the $\alpha\rho^\mu$ term in (10.21) will limit to zero as ε limits to zero because μ is positive. Meanwhile, the $\beta\rho^{-\mu'}$ term in (10.21) will be absent ($\beta = 0$) if $\mu' \geq \frac{3}{2}$ because if that isn't the case, then the presence of the $\beta\rho^{-\mu'}$ term with $\mu' \geq \frac{3}{2}$ runs afoul of the constraint in (10.19). Since μ' is no smaller than $\frac{3}{2}$ when $\lambda \geq \frac{3}{4}$, the issue to address is whether a $(-\Delta_2)$ -eigensection from $\mathbb{H}_p$ with eigenvalue less than $\frac{3}{4}$ can appear in (10.17).

To see about a lower bound for those λ_j : The key input for the analysis of those eigenvalues is the fact that each edge of Γ is invariant under the action of an order 3 subgroup of G that acts freely on the normal disks to interior points of that edge. That subgroup for an incident edge to p acts as an order three rotation on the constant radius spheres centered at the vertex p . With this understood, fix a point (to be denoted by q) from the configuration $\mathfrak{p}$. The order 3 subgroup of G that fixed $\mathfrak{p}$ acts to permute the other points in $\mathfrak{p}$. Each $\mathfrak{f}_j$ that appears in (10.17) must be invariant with regards to this $\mathbb{Z}/3\mathbb{Z}$ subgroup which implies in turn that $|\mathfrak{f}_j|$ near q is bounded by $c_0 \operatorname{dist}(\cdot, q)^{3/2}$ (See [26, Proposition 2.2, Equation 4.6]).

To explore the ramifications of this observation, introduce by way of notation ∇_2 to denote the metric covariant derivative on sections of T^*S^2 . The section $d\mathfrak{f}_j$ of $T^*S^2 \otimes \mathcal{I}$ obeys the differential equation

$$(10.22) \quad \nabla_2^\dagger \nabla_2(d\mathfrak{f}_j) + d\mathfrak{f}_j = \lambda_j d\mathfrak{f}_j.$$

Having fixed $\varepsilon > 0$ but tiny, let

$$(10.23) \quad \chi_\varepsilon = \chi(1 - \varepsilon^{-1} \operatorname{dist}(\cdot, \mathfrak{p})),$$

where

$$(10.24) \quad \operatorname{dist}(\cdot, \mathfrak{p}) := \min_{p \in \mathfrak{p}} \operatorname{dist}(\cdot, p).$$

To be sure, this function has value 1 where the distance to $\mathfrak{p}$ is greater than ε , and it has value zero where the distance to $\mathfrak{p}$ is less than $\varepsilon/4$. Take the inner product of both sides of the equation in (10.22) with $\chi_\varepsilon d\mathfrak{f}_j$ and then integrate the result over S^2 . Since χ_ε is zero near $\mathfrak{p}$, the result after an

integration by parts leads to this:

$$(10.25) \quad \int_{S^2} \chi_\varepsilon (|\nabla_2 d\mathfrak{f}_j|^2 + |d\mathfrak{f}_j|^2) \leq \lambda_j \int_{S^2} |d\mathfrak{f}_j|^2 + \mathfrak{e},$$

with $\mathfrak{e}$ being the term from the integration by parts with derivatives on χ_ε . In particular, the norm of $\mathfrak{e}$ obeys the bound

$$(10.26) \quad |\mathfrak{e}| \leq c_0 \varepsilon^{-2} \int_{S^2} \chi_{\varepsilon/8} (1 - \chi_{4\varepsilon}) |d\mathfrak{f}_j|^2.$$

To see about the integral on the right in (10.26), multiply both sides of the eigenvalue equation $-\Delta_2 \mathfrak{f}_j = \lambda_j \mathfrak{f}_j$ by $\chi_{\varepsilon/8} (1 - \chi_{4\varepsilon}) \mathfrak{f}_j$ and then integrate the result. An integration by parts leads in turn to the inequality below:

$$(10.27) \quad \int_{S^2} \chi_{\varepsilon/8} (1 - \chi_{4\varepsilon}) |d\mathfrak{f}_j|^2 \leq c_0 \varepsilon^{-2} \int_{S^2} (1 - \chi_{8\varepsilon}) |\mathfrak{f}_j|^2.$$

This last inequality leads to a $c_0 \varepsilon^3$ bound for the integral over S^2 of $\int_{S^2} \chi_{\varepsilon/8} (1 - \chi_{4\varepsilon}) |d\mathfrak{f}_j|^2$ since $|\mathfrak{f}_j| < c_0 \varepsilon^{3/2}$ on the domain of the integral on the left-hand side of (10.26). That bound with (10.26) implies in turn that $|\mathfrak{e}| \leq c_0 \varepsilon$ which has limit zero as $\varepsilon \rightarrow 0$. With this understood, take ε to zero in (10.25) to see that λ_j must be greater than 1. (Note that in this regards that $d\mathfrak{f}_j$ can't be identically zero since $\mathcal{I}$ is not the trivial line bundle.)

Part 5: To summarize where things stand at this point: Parts 2–4 have established that the $T^*S^3 \otimes \mathcal{I}$ valued 1-form $y \equiv \psi - \frac{1}{\mathbb{E}} d\phi$ is in the $T^*S^3 \otimes \mathcal{I}$ version of the Hilbert space $\mathbb{H}$. This 1-form is coclosed, and it obeys (10.2). It is also G -invariant. This part of the proof and Part 6 explain why $*dy$ is also in this same version of $\mathbb{H}$. This part focuses on the behavior $*dy$ near interior points of the edges of Γ . The next part of the proof considers $*dy$ near the vertices of Γ .

Supposing that p denotes a point in the interior of an edge of Γ , reintroduce the coordinates (z, t) from Part 2 of this section for a neighborhood of p in S^3 . (These are defined just prior to (10.3).) Thus, z is a complex coordinate and t is $\mathbb{R}$ -valued; and both are zero at the point p . With regards to (10.3), keep in mind that $\rho = |z|$. The metric in (10.3) when written using z and t has the form below:

$$(10.28) \quad dz \otimes d\bar{z} + dt^2 + \frac{1}{1 - |z|^2} (z d\bar{z} + \bar{z} dz)^2 + |z|^2 dt^2.$$

Now write y using the coordinate basis as

$$(10.29) \quad y = w_t dt + w dz + \bar{w} d\bar{z}.$$

(This is the depiction in (5.5) if w is written using its real and imaginary parts as $w = w_1 - iw_2$.)

Let β_p denote for the moment the $r = \frac{1}{8192c} r_p$ version of $\chi \left(\frac{\text{dist}(\cdot, p)}{r} - 1 \right)$ which is equal to 1 near p and equal to zero where $\text{dist}(\cdot, p)$ is greater than $\frac{1}{4096c} r_p$. Lemma 6.2 guarantees that $\beta_p \nabla_t y$ is in $\mathbb{H}$. As will be explained

momentarily, the fact that y is G -invariant implies that $\beta_p dw_t$ is also in $\mathbb{H}$. Granted these last two points, it then follows that $|\nabla(*dy)|^2$ has finite integral on the radius $\frac{1}{2 \times 8192c} r_p$ ball centered at p if this is so for $|\nabla(\partial w / \partial \bar{z})|^2$. For the latter, this follows algebraically if $\beta_p dw_t$ and $\beta_p \nabla_t y$ are in $\mathbb{H}$ using the two identities $*d(*dy) = \mathbf{E}y$ and $d^*y = 0$.

To see about dw_t , the first key point is that w_t in the radius $\frac{1}{1024c} r_p$ ball centered at p obeys a differential equation that has the schematic form below:

$$(10.30) \quad -\widehat{\Delta} w_t + 2z \frac{\partial}{\partial t} w + 2\bar{z} \frac{\partial}{\partial t} \bar{w} = \mathbf{E} w_t,$$

with $\widehat{\Delta}$ denoting the operator depicted below (the notation reintroduces the nonnegative function ρ and the $\mathbb{R}/2\pi\mathbb{Z}$ valued function θ by writing z as $\rho e^{i\theta}$):

$$(10.31) \quad \widehat{\Delta} = \frac{1 - \rho^2}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{1 - \rho^2} \frac{\partial^2}{\partial t^2}.$$

Fourier series (mimicking what was done in Part 2) will be used with (10.30) to analyze the behavior of w_t . To this end, reintroduce from Part 2 the product structure for $\mathcal{I}$ where $\theta \in (-\pi, \pi)$. Having done that, write

$$(10.32) \quad \beta_p w_t = \sum u_{k,n}(\rho) e^{i(k+\frac{1}{2})\theta} e^{i1024\pi c n t / r_p}.$$

Introduce by way of notation,

$$(10.33) \quad \mathfrak{k} \equiv \beta_p \left(2z \frac{\partial}{\partial t} w + 2\bar{z} \frac{\partial}{\partial t} \bar{w} \right)$$

and write $\mathfrak{k}$ similarly as

$$(10.34) \quad \mathfrak{k} = \sum \chi_{k,n}(\rho) e^{i(k+\frac{1}{2})\theta} e^{i1024\pi c n t / r_p}.$$

Note in particular: The fact that $\beta_p \nabla_t y$ is in $\mathbb{H}$ implies that

$$(10.35) \quad |\chi_{k,n}| \leq c_0 \rho.$$

Now it follows from (10.35) and (10.31) that $u_{k,n}$ can be written as

$$(10.36) \quad u_{k,n} = (\alpha_{k,n} + \rho^2 \mathfrak{f}_{k,n}) \rho^{k+\frac{1}{2}} + (\beta_{k,n} + \rho^2 \mathfrak{h}_{k,n}) \rho^{-(k+\frac{1}{2})} + \mathfrak{e}_{k,n},$$

where $|\mathfrak{e}_{k,n}| \leq c_0 \rho^3$ and where $|\mathfrak{f}_{k,n}|$ and $|\mathfrak{h}_{k,n}|$ are bounded functions of ρ . Because w_t is in $\mathbb{H}$ and it is G -invariant, an argument essentially a verbatim copy of that used subsequent to (10.6) can be used to deduce that $\alpha_{k,n} = 0$ for $k \leq 0$ and that $\beta_{k,n} = 0$ for $k \geq -1$. Thus,

$$(10.37) \quad |u_{k,n}| \leq c_0 z_{k,n} \rho^{3/2}.$$

With this bound in hand, the arguments from Part 3 can be copied almost verbatim to see that $\beta_p \nabla w_t$ is in $\mathbb{H}$ as claimed.

Part 6: This part of Section 10 and the final part explain why $|\nabla(*dy)|^2$ is integrable near the vertices of Γ . To keep the notation succinct, let q now

denote $*dy$. Keep in mind from Part 5 that $|\nabla q|^2$ is integrable on sufficiently small radius balls centered at the interior points to the edges of Γ .

With the preceding in mind, let ρ now denote the function on the $\frac{1}{10^6 c} r_p$ ball centered at p that gives the distance to p . Introduce as before, a Gaussian coordinate system centered at p to identify this ball centered at p with the same radius ball centered at the origin in $\mathbb{R}^3$. Use ρ now to denote the radial coordinate on $\mathbb{R}^3$. Let S^2 again denote the $\rho = 1$ sphere in $\mathbb{R}^3$ and let $\mathfrak{p} \subset S^2$ denote the intersection between S^2 and the radial projection to S^2 of the intersection of Γ with any constant $\rho \in (0, \frac{1}{10^6 c} r_p)$ sphere. (The graph Γ in the $\rho < \frac{1}{10^6 c} r_p$ ball is the intersection of that ball with the cone on the configuration $\mathfrak{p}$.)

Let q_ρ denote the $d\rho$ component of q on that $\rho < \frac{1}{10^6 c} r_p$ ball (this is the pairing between q and the dual vector field $\frac{\partial}{\partial \rho}$). The rest of this part of the proof focuses on q_ρ and in so doing, it proves that $|dq_\rho|^2$ is integrable on small ρ balls.

This q_ρ is a section of $\mathcal{I}$ on the complement of Γ in that ball. By virtue of q obeying (10.2), this section q_ρ obeys the equation below:

$$(10.38) \quad -\frac{1}{\sin^2 \rho} \frac{d^2}{d\rho^2} (\sin^2 \rho q_\rho) - \frac{1}{\sin^2 \rho} \Delta_2 q_\rho = E q_\rho,$$

with Δ_2 denoting (as before) the standard Laplacian on the S^2 . To see what this equation requires of q_ρ , expand q_ρ as was done with ς in (10.17) using the $\mathbb{H}_\mathfrak{p}$ eigensections of Δ_2 :

$$(10.39) \quad q_\rho = \sum \mathfrak{b}_j(\rho) \mathfrak{f}_j.$$

(Remember in this regard that $\{\mathfrak{f}_1, \mathfrak{f}_2, \dots\}$ is an $\mathbb{L}_\mathfrak{p}$ -orthonormal basis of eigensections of Δ_2 from $\mathbb{H}_\mathfrak{p}$. In what follows λ_j for $j \in \{1, 2, \dots\}$ denotes the eigenvalue of the corresponding $\mathfrak{f}_j$.) Of particular note is that the sum in (10.39) is convergent in the $\mathbb{L}$ -topology on the radius $\frac{1}{10^6 c} r_p$ ball centered at the origin in $\mathbb{R}^3$. This is to say that

$$(10.40) \quad \lim_{N \rightarrow \infty} \int_0^{\frac{r_p}{10^6 c}} \left| q_\rho - \sum^N \mathfrak{b}_j \mathfrak{f}_j \right|^2 \sin^2 \rho d\rho = 0.$$

where the notation here and below has $\sum^N$ signifying that the sum is over the indices j from the set $\{1, \dots, N\}$.

By virtue of (10.38), any given $\mathfrak{b}_j$ obeys the equation below:

$$(10.41) \quad -\frac{1}{\sin^2 \rho} \frac{d^2}{d\rho^2} (\mathfrak{b}_j \sin^2 \rho) + \frac{1}{\sin^2 \rho} \lambda_j \mathfrak{b}_j = E \mathfrak{b}_j.$$

This equation requires in turn that $\mathfrak{b}_j$ have the form

$$(10.42) \quad \mathfrak{b}_j = \alpha \rho^\mu (1 + \rho^2 \mathfrak{s}) + \beta \rho^{-\mu'} (1 + \rho^2 \mathfrak{t}),$$

with α and β being constant, with μ and μ' being the numbers depicted below in (10.43) and with $\mathfrak{s}$ and $\mathfrak{t}$ being functions of ρ with bounded norm

(which can depend on j):

$$(10.43) \quad \mu = \left(-\frac{3}{2} + \frac{1}{2}(1 + 4\lambda_j)^{1/2}\right) \quad \text{and} \quad \mu' = \left(\frac{3}{2} + \frac{1}{2}(1 + 4\lambda_j)^{1/2}\right).$$

With regards to β : The number β must be zero because if β is non-zero, then the $\psi = y$ and $x = p$ version of the integral that is depicted in the fourth bullet of (6.3) would diverge, because μ' is greater than 2 since λ_j is positive.

With regards to μ : As explained directly, μ is greater than -0.4 . This is because q_ρ is invariant with respect to the $\mathbb{Z}/3\mathbb{Z}$ subgroup of G that fixes any given edge incident to p . Given such an edge, this $\mathbb{Z}/3\mathbb{Z}$ subgroup acts as a rotation of S^2 that fixes the corresponding point in $\mathfrak{p}$. The $\mathfrak{f}_j$ must be invariant with respect to these subgroups, which implies (as explained in [26]) that $|\mathfrak{f}_j|$ is bounded by a constant multiple of $\text{dist}(\cdot, p)^{3/2}$. That implies in turn that $d\mathfrak{f}_j$ is from the $T^*S^2 \otimes \mathcal{I}$ version of $\mathbb{H}_{\mathfrak{p}}$ which implies in turn (as noted previously) that λ_j is strictly greater than 1 (see (10.25) and the subsequent remarks).

Any $r \in (0, \frac{1}{10^6 c} r_p)$ version of the bound below follows directly since $\mu > -0.4$ (this bound has c_j being independent of r):

$$(10.44) \quad \int_0^r |\mathfrak{b}_j|^2 d\rho < c_j r^{0.2}.$$

The preceding bounds with the bounds implied by the $\varepsilon \rightarrow 0$ limit of (10.25) can be used to see that $|\nabla q_\rho|^2$ is integrable near p . This is done using (10.38) by first approximating q_ρ using a sequence of finite sum versions of (10.39) with the N 'th element in this sequence (for $N = 1, 2, \dots$) being $\Sigma^N \mathfrak{b}_j \mathfrak{f}_j$. Let $q_{\rho, N}$ denote this approximation. A key fact now is that $q_{\rho, N}$ obeys (10.38) also. Granted that, fix ε to be positive but very small. Letting r denote the $\frac{1}{4 \times 10^6 c} r_p$, multiply both sides of the $q_{\rho, N}$ version of (10.38) by $\chi(1 - \rho/\varepsilon)\chi(\rho/r - 1)q_{\rho, N} \sin^2 \rho$. Having done that, then integrate and use integration by parts to see that

$$(10.45) \quad \int_{B_r - B_\varepsilon} |\nabla q_{\rho, N}|^2 \leq c_0 \varepsilon^{-2} \int_{B_{2\varepsilon}} |q_{\rho, N}|^2 + c_0 r^{-2} \int_{B_{2r} - B_r} |q_{\rho, N}|^2.$$

(Note that the derivation of this bound uses $\varepsilon = 0$ version of (10.25) and the fact that the eigenvalues λ_j for the $\mathfrak{f}_j$'s that appear in (10.39) are greater than 1.) To finish from here: The $r = 2\varepsilon$ version of (10.44) implies that the leftmost term on the right-hand side of (10.45) has an $\varepsilon \rightarrow 0$ limit; and also that this limit is 0. The latter fact and the dominated convergence theorem leads from (10.45) and (10.40) to an N -independent bound for the integral of $|\nabla q_{\rho, N}|^2$ on B_r . And, much the same manipulations can be used to conclude that $\{\chi(\frac{\rho}{r} - 1) \nabla q_{\rho, N}\}_{N=1, 2, \dots}$ is a Cauchy sequence in the B_r version of the Hilbert space $\mathbb{H}_B$. That last fact implies that $\chi(\frac{\rho}{r} - 1) q_\rho$ is in $\mathbb{H}_B$ too.

Part 7: The fact that $\chi\left(\frac{\rho}{r} - 1\right) q_\rho$ is in $\mathbb{H}_B$ will be used momentarily to prove that the bound below in (10.46) holds for any given value of $s \leq \frac{1}{10^8 c} r_p$ with c_* independent of s .

$$(10.46) \quad \int_{B_s} \frac{1}{\rho^2} |q|^2 \leq c_*.$$

Suppose for the moment that (10.46) holds. As explained directly, that bound implies that the whole of $\chi\left(\frac{\rho}{r} - 1\right) q$ is in the $T^*S^3 \otimes \mathcal{I}$ version of $\mathbb{H}_B$. This is done by first noting that (6.5) holds with $\psi = q$ since q obeys $\nabla^\dagger \nabla q + \text{Ric} \cdot q = \text{Eq}$ (because $d^*q = 0$ and $*d(*dq) = \text{Eq}$). Keeping the latter fact in mind, fix $\varepsilon > 0$ but much less r (which is $\frac{1}{4 \times 10^6 c} r_p$) and multiply both sides of the $\psi = q$ version of (6.5) by $\chi\left(1 - \frac{\rho}{\varepsilon}\right) \chi\left(\frac{\rho}{r} - 1\right)$. Then integrate the result over the radius $2r$ ball centered at p and integrate by parts in the resulting identity to obtain the inequality

$$(10.47) \quad \int_{B_r - B_\varepsilon} |\nabla q|^2 \leq c_0 \varepsilon^{-2} \int_{B_{2\varepsilon}} |q|^2 + c_0 r^{-2} \int_{B_{2r}} |q|^2.$$

(The integration by parts does not get hung up on the edges of Γ in B_{2r} because $|\nabla q|^2$ is integrable near interior points of the edges.)

The right-hand side of (10.47) is bounded as ε limits to zero by virtue of the $s = 2\varepsilon$ version of (10.46). The existence of that limit implies that $|\nabla q|^2$ is integrable on B_r and that in turn implies (with another appeal to (10.46)) the desired conclusion that $\chi\left(\frac{\rho}{r} - 1\right) q$ is in the $B = B_r$ version of $\mathbb{H}_B$. (That other appeal is used in a straightforward way to see that $\chi\left(1 - \frac{\rho}{\varepsilon}\right) \chi\left(\frac{\rho}{r} - 1\right) q$ converges in $\mathbb{H}_B$ as ε limits to zero.)

With the preceding understood, the subsequent paragraphs derive (10.46). To this end, note first that (10.46) is obeyed with q replaced by q_ρ because $\chi\left(\frac{\rho}{r} - 1\right) q_\rho$ is in $\mathbb{H}_B$ and thus the $\psi = \chi\left(\frac{\rho}{r} - 1\right) q_\rho$ version of (4.2) holds.

The rest of q is $q - q_\rho d\rho$; this is written in what follows as $\hat{q}$ and it is viewed as a ρ -dependent, $\mathcal{I}$ -valued 1-form on $S^2 - \mathbf{p}$. (To be sure about what this means: At any given value of ρ , the norm of $\hat{q}$ as an $\mathcal{I}$ -valued section of T^*S^2 is $\sin \rho$ times its norm as an $\mathcal{I}$ -valued section of T^*S^3 .) The $d\rho$ component of the identity $*dq = \text{Eq}$ can be depicted using $\hat{q}$ in the manner of the leftmost equation in (10.48). The rightmost equation in (10.48) uses $\hat{q}$ to depict the identity $d^*q = 0$. The notation in (10.48) uses $\hat{d}$ and $\hat{*}$ to denote the respective exterior derivative along S^2 and the Hodge star on S^2 . Also, (10.48) uses y_ρ to denote the $d\rho$ component of y .

$$(10.48) \quad \frac{1}{\sin^2 \rho} \hat{*} \hat{d} \hat{q} = \text{E} y_\rho \quad \text{and} \quad \frac{1}{\sin^2 \rho} \hat{*} (\hat{d} \hat{*} \hat{q}) = -\frac{\partial}{\partial \rho} q_\rho - \frac{2 \cos \rho}{\sin \rho} q_\rho.$$

Now comes the first key observation: The square of the norm of the right-hand side of both equations in (10.48) is integrable on B_r . As a consequence,

there exists $c_{\dagger}$ such that for any given small but positive ε ,

$$(10.49) \quad \int_{B_r - B_\varepsilon} \frac{1}{\sin^4 \rho} \left(|\hat{d}\hat{q}|^2 + |\hat{d} \star \hat{q}|^2 \right) < c_{\dagger}.$$

Now comes the second key observation: Since $|\nabla q|^2$ is integrable near points in the interior of the edges of Γ , the $\mathcal{I}$ -valued 1-form $\hat{q}$ at all but a measure zero set of ρ values is in the $T^*S^2 \otimes \mathcal{I}$ version of the Hilbert space $\mathbb{H}_p$ (which implies that the square of the norm of its S^2 -covariant derivative is integrable on S^2 at these values of ρ). That implies in turn that some integration by parts can be used (with no hang-up on the edges of Γ in $B_r - B_\varepsilon$) to write (10.49) as done below (with $\hat{\nabla}$ denoting the round metric's covariant derivative and with $|\cdot|_{S^2}$ denoting the round metric's norm on 1-forms and tensors).

$$(10.50) \quad \int_{B_r - B_\varepsilon} \frac{1}{\sin^4 \rho} \left(|\hat{\nabla} \hat{q}|_{S^2}^2 + |\hat{q}|_{S^2}^2 \right) < c_{\dagger}.$$

Because the S^3 norm of $\hat{q}$ is $\frac{1}{\sin \rho} |\hat{q}|_{S^2}$, this last inequality leads directly to (10.46) with q replaced by $\hat{q}$ by taking ε ever smaller with limit zero.

Part 2. Polytope symmetry, equivariance, and topological invariants

11. Regular 4-polytopes and their symmetries

Part I develops the analytic framework for $\mathbb{Z}/2$ -harmonic objects on $S^3 \setminus \Gamma$ with values in a real line bundle $\mathcal{I} \rightarrow S^3 \setminus \Gamma$. Part II verifies the geometric properties needed to apply the analytic results of Part I to the graphs $\Gamma \subset S^3$ obtained from regular 4-polytopes. Concretely, we will need (i) an explicit description of the induced action of $G = \text{Sym}^+(P)$ on $S^3 \setminus \Gamma$ and its compatibility with the $\{\pm 1\}$ -local system $\mathcal{I}$, (ii) the existence of odd-order cyclic subgroups fixing edges of Γ pointwise acting by rotations on the normal 2-planes, and (iii) a splitting of the lift extension for the G -action on the associated connected double cover, yielding fiberwise linear automorphisms of $\mathcal{I}$.

This section develops the background for Part II. Section 11.1 records the quaternionic model for $\text{SO}(4)$, used to write elements of G explicitly and to compute their fixed-point sets, vertex actions, and facet actions. Sections 11.2 and 11.3 recall the regular 4-polytopes, their chambers, radial projections, Schläfli symbols, and local link/figure conventions. Finally, Sections 11.4 and 11.5 extract the structural consequences used later: even valency of the radial graph, odd-order cyclic subgroups fixing edges pointwise, liftability of the G -action to $\mathcal{I}$, and standard presentations of the relevant rotation subgroups.

11.1. Quaternionic representation of $\text{SO}(4)$ and the G -action.

Since each regular 4-polytope $P \subset \mathbb{R}^4$ is centered at the origin, its rotational symmetry group $G = \text{Sym}^+(P) \subset \text{SO}(4)$ acts linearly on $\mathbb{R}^4$ and hence

on S^3 . It is convenient to work in the quaternionic model $\mathbb{R}^4 \cong \mathbb{H}$, where elements of $SO(4)$ are represented (up to sign) by pairs in $S^3 \times S^3$ and many fixed-set computations reduce to quaternion algebra. Let

$$\mathbb{H} = \{w + xi + yj + zk : w, x, y, z \in \mathbb{R}\}, \quad i^2 = j^2 = k^2 = ijk = -1.$$

Write $\Re(p) = w$ and $\Im(p) = xi + yj + zk$, so that $\mathbb{H} = \mathbb{R} \oplus \Im\mathbb{H}$ with $\Im\mathbb{H} \cong \mathbb{R}^3$. Quaternion conjugation is $\bar{p} = w - xi - yj - zk$ and the norm is

$$|p|^2 = p\bar{p} = w^2 + x^2 + y^2 + z^2.$$

We identify $\mathbb{R}^4 \cong \mathbb{H}$ by

$$(x_1, x_2, x_3, x_4) \longleftrightarrow x_1 + x_2i + x_3j + x_4k,$$

so that $\langle p, q \rangle = \Re(p\bar{q})$, $|p|^2 = p\bar{p}$. In particular, the quaternion norm agrees with the Euclidean norm.

The Spin(4) covering of $SO(4)$. Let $S^3 = \{q \in \mathbb{H} : |q| = 1\} \cong \mathrm{SU}(2)$. Define

$$\Pi : S^3 \times S^3 \rightarrow SO(4), \quad \Pi(q_L, q_R)(p) = q_L p q_R^{-1}.$$

Then Π is a surjective homomorphism with kernel $\{(1, 1), (-1, -1)\} \cong \mathbb{Z}_2$; hence $SO(4) \cong (S^3 \times S^3) / \{\pm(1, 1)\}$. Indeed, Π is a group homomorphism because $\Pi(q_L, q_R) = L_{q_L} R_{q_R^{-1}}$ and composition corresponds to quaternion multiplication. If $\Pi(q_L, q_R) = \mathrm{id}$, then $q_L p q_R^{-1} = p$ for all p , whence $q_L = q_R$ and q_L commutes with i, j, k , so $q_L = \pm 1$. Surjectivity follows from $\mathrm{Spin}(4) = S^3 \times S^3$ being the universal cover of $SO(4)$. Thus,

$$SO(4) \cong \frac{\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R}{\{(1, 1), (-1, -1)\}}.$$

Thus, every $g \in SO(4)$ can be represented (nonuniquely) by a pair $(q_L, q_R) \in S^3 \times S^3$, unique up to the simultaneous sign change $(q_L, q_R) \mapsto (-q_L, -q_R)$ [20, §2.7, p. 42].

Left/right multiplication and the special case Ad_q . Identifying $\mathbb{H} \cong \mathbb{R}^4$ via the ordered basis $(1, i, j, k)$, left and right multiplication are $\mathbb{R}$ -linear maps

$$L_q(p) = qp, \quad R_q(p) = pq.$$

If $|q| = 1$ then $L_q, R_q \in SO(4)$. An important special case is $q_L = q_R = a \in S^3$, where $\mathrm{Ad}_a(x) := \Pi(a, a)(x) = axa^{-1}$. More precisely, suppose $a \notin \{\pm 1\}$, so write $a = \cos \theta + u \sin \theta$ with $u \in \Im\mathbb{H}$, $|u| = 1$, and $0 < \theta < \pi$. For $x \in u^\perp \subset \Im\mathbb{H}$, one has $ux = -xu$, and

$$\mathrm{Ad}_a(x) = axa^{-1} = \cos(2\theta)x + \sin(2\theta)ux.$$

Since $x \mapsto ux$ is the quarter-turn on $u^\perp$, this is rotation by angle 2θ about the axis $\mathrm{Span}_{\mathbb{R}}\{u\}$ in $\Im\mathbb{H}$. Also, $\mathrm{Ad}_a(1) = 1$. Consequently, for $a \notin \{\pm 1\}$, the fixed-point set of Ad_a in $\mathbb{R}^4 \simeq \mathbb{H} = \mathbb{R} \oplus \Im\mathbb{H}$ is the 2-plane $\mathrm{Span}_{\mathbb{R}}\{1, u\}$.

In the exceptional cases $a = \pm 1$, Ad_a is the identity and its fixed-point set is all of $\mathbb{R}^4$.

11.2. Regular 4-polytopes, chambers, and barycentric subdivision. This section introduces definitions leading to convex regular 4-polytopes, whose 1-skeleta supply local models for $\mathbb{Z}/2$ harmonic forms. Regular 4-polytopes are defined in terms of chambers. The following exhibits some basic notions.

DEFINITION 11.1 (Convex polytope [6, Def. A.1.1]). *A convex polytope P in an affine space $\mathbb{A}$ is the convex hull of a finite subset of $\mathbb{A}$. Its dimension is the dimension of the affine subspace it spans.*

DEFINITION 11.2 (Supporting hyperplane [6, p. 402, Appendix A.1]). *Suppose X is a closed convex subset of $\mathbb{A}$. An affine hyperplane H is a supporting hyperplane of X if $X \cap H \neq \emptyset$ and if X is contained in one of the half-spaces bounded by H . (That is to say, X lies on one side of H .)*

DEFINITION 11.3 (Face of a polytope [6, p. 402, Appendix A.1]). *Suppose P is a convex polytope and H a supporting hyperplane. Then $P \cap H$ is also a convex polytope, called a face of P . A proper face is a face in the set of all nonempty faces of a convex polytope P other than P itself. The collection of all proper faces ∂P is the boundary complex of P .*

DEFINITION 11.4 (Chain [6, p. 126, Sec. 7.2]). *For any poset $\mathcal{P}$, a chain is a nonempty totally ordered subset of $\mathcal{P}$. A chain not properly contained in a larger chain is called a maximal chain.*

DEFINITION 11.5 (Barycentric subdivision [6, Appendix A.3, p. 411]). *Let P be a convex polytope. Choose for each face $F \leq P$ a point v_F in the relative interior of F . For each chain $\alpha = (F_1 < \cdots < F_k)$ of faces of P , let $\hat{\alpha}$ be the simplex spanned by $v_{F_1}, \dots, v_{F_k}$. The simplices $\{\hat{\alpha}\}$ form a simplicial complex bP , called the barycentric subdivision of P .*

Non-maximal chains give the lower-dimensional faces of the barycentric subdivision. For example, in the barycentric subdivision of the boundary of a 4-polytope, a maximal chain $V < E < F < C$ gives a tetrahedral chamber $[v_V, v_E, v_F, v_C]$, while a shorter chain such as $V < E < F$ gives the triangular face $[v_V, v_E, v_F]$, and a chain $E < C$ gives the edge $[v_E, v_C]$. Applied to the boundary complex ∂P , this construction gives the barycentric subdivision $b(\partial P)$; see Fig. 2. Its top-dimensional simplices are precisely the chambers.

DEFINITION 11.6 (Chamber [6, p. 94]). *A chamber of a convex polytope P^n is a maximal chain of proper faces of P , $F_0 < F_1 < \cdots < F_{n-1}$, where $\dim F_i = i$. Equivalently, a chamber is a top-dimensional simplex of $b(\partial P)$, and such a simplex is identified with a maximal chain $\{F_0 < F_1 < \cdots < F_{n-1}\}$ of faces of ∂P .*

Thus, the chambers come from maximal chains, and the full barycentric subdivision is obtained by including all chains.

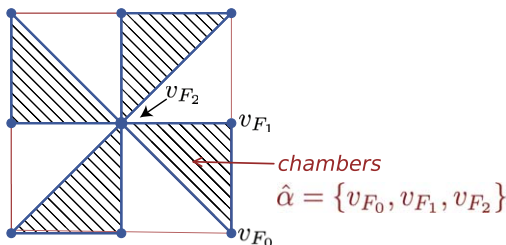


FIGURE 2. Illustration of barycentric subdivision $b(\partial P)$ of a two-dimensional boundary complex of a 3-dimensional polytope P . For a maximal chain of proper faces $\alpha = (F_0 < F_1 < F_2)$ in ∂P , the simplex $\hat{\alpha} = [v_{F_0}, v_{F_1}, v_{F_2}]$ is a top-dimensional simplex of $b(\partial P)$, i.e. a chamber.

DEFINITION 11.7 (Regular polytope [6, p. 94]; [6, Def. B.2.1, p. 424]). *A polytope in Euclidean space is regular if its symmetry group $\text{Isom}(P)$ acts transitively on the set of chambers, which is denoted $\text{Cham}(b(\partial P))$.*

Recall the radial projection of the 1-skeleton of a regular 4-polytope is defined as

$$\Gamma = \left\{ \frac{x}{|x|} : x \in P^{(1)} \setminus \{0\} \right\} \subset S^3.$$

For each polytope edge $[u, v]$ of a centered regular polytope, the radial image of $[u, v]$ is contained in $\text{span}_{\mathbb{R}}\{u, v\} \cap S^3$. Since $0 \notin [u, v]$, the endpoints u and v are not antipodal, so $\text{span}_{\mathbb{R}}\{u, v\}$ is a two-plane and its intersection with S^3 is a great circle. Thus, the radial image of $[u, v]$ is a geodesic arc in S^3 . This verifies the geodesic-edge hypothesis used in Part I, most directly in the local edge estimates of Lemmas 6.1, 6.2, and 6.4, and hence in the proof of Lemma 4.2.

11.3. Links, edge figures, Schläfli symbols and local models.

Schläfli symbols record the local incidence data of a regular 4-polytope in a way compatible with the chamber structure of the barycentric subdivision. Schläfli symbols play an essential role in determining the geometric and topological requirements on polytopes prescribed by Section 3, and also provide a description of the symmetry group of regular polytopes. We use them to encode vertex figures, edge figures, and the cyclic order of facets around an edge.

We first recall the abstract notion of a link and its geometric realization. For a simplex τ of an abstract simplicial complex Δ , the link is

$$\text{Lk}_{\Delta}(\tau) := \{\sigma \in \Delta \mid \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Delta\}$$

[18, Def. 2.13, p. 11]. Although $\text{Lk}_{\Delta}(\tau)$ is defined abstractly as a collection of simplices, it has a geometric realization obtained by replacing each abstract simplex by the corresponding Euclidean simplex and gluing along common faces. When $\tau = \{v\}$ is a vertex and the complex is embedded, this

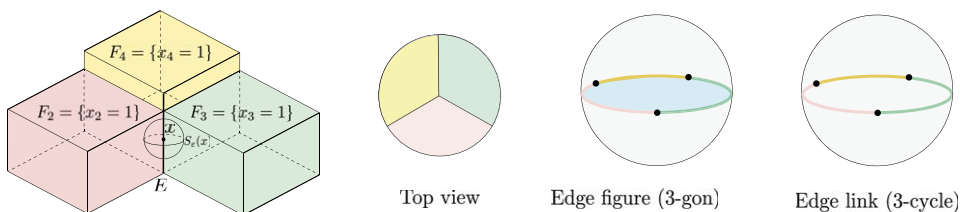


FIGURE 3. Local model near an interior point $x \in E^\circ$ of an edge E (Eg. 11.9). The edge is contained in the three cubic facets F_2 , F_3 , and F_4 . A small sphere $S_\varepsilon(x)$ registers the local transverse picture. The top view records the three sectors determined by the incident facets; the edge figure is the triangular 2-cell bounded by that cycle; and the edge link is the equatorial 3-cycle.

geometric realization of $\text{Lk}_\Delta(v)$ is naturally homeomorphic to the intersection of the embedded complex with a sufficiently small sphere centered at v [18, p. 27]. For example, if Δ is the boundary complex of a tetrahedron with vertices 1, 2, 3, 4, then $\text{Lk}_\Delta(\{1\})$ has vertices $\{2\}, \{3\}, \{4\}$ and edges $\{2, 3\}, \{2, 4\}, \{3, 4\}$; its geometric realization is the boundary of a triangle, matching the intersection of the tetrahedron boundary with a sufficiently small sphere centered at vertex 1.

For a general face σ , choose a point $x \in \sigma^\circ$ and set $d = \dim \sigma$. A neighborhood of x in the complex is locally modeled as $B^d \times Q$, where B^d is a small d -dimensional ball in the directions tangent to σ , while Q is the transverse complex obtained by slicing normal to σ . The point x corresponds to the central vertex of Q , and the link of σ is the link of this central vertex in Q [18, p. 27]. In particular, for an edge $E \subset \partial P$, the factor B^1 is the direction along E , and Q records how the incident faces or facets meet after this edge direction is removed. The edge link is the boundary of the edge figure: for a 3-polytope, such as the ordinary cube, it is the two endpoints of an interval; for a 4-polytope, such as the 5-cell, it is a polygonal cycle. Thus, when r facets meet cyclically around an edge of a 4-polytope, the edge link is an r -cycle and the edge figure is the r -gon bounded by it; see Examples 11.8, 11.9, and Fig. 3.

EXAMPLE 11.8. Let $P = [0, 1]^3$ be the ordinary cube and let

$$E = \{(t, 1, 1) : 0 \leq t \leq 1\}$$

be an edge. Choose $x = (t_0, 1, 1) \in E^\circ$. Near x , the boundary complex ∂P is locally $B^1 \times Q$, where B^1 is a small interval in the E -direction and Q is the transverse V-shaped complex obtained by slicing with the plane $\{x_1 = t_0\}$. The two rays of Q come from the two square faces incident to E . The link of the central vertex of Q consists of two points. These two points form the edge link, and the edge figure is the interval joining them.

EXAMPLE 11.9. Let $P = [0, 1]^4$ be the 8-cell and let

$$E = \{(t, 1, 1, 1) : 0 \leq t \leq 1\}.$$

Choose $x = (t_0, 1, 1, 1) \in E^\circ$. Near x , the boundary complex ∂P is locally $B^1 \times Q$, where B^1 is a small interval in the E -direction and Q is the transverse complex obtained by slicing with the hyperplane $\{x_1 = t_0\}$. The three sectors of Q come from the three cubic facets incident to E . The link of the central vertex of Q is a 3-cycle. This cycle is the edge link, and the edge figure is the associated 3-gon, see Fig. 3.

11.3.1. *Schläfli symbols and edge figures.* For a regular 4-polytope with Schläfli symbol $\{p, q, r\}$, each 2-face is a p -gon, and each facet is a regular polyhedron of type $\{p, q\}$; within each facet, exactly q of these 2-faces meet at each vertex. Exactly r facets meet cyclically around each edge. Equivalently, the edge link is an r -cycle, and the edge figure is the polygonal 2-cell bounded by this cycle. Fig. 3 shows the case $r = 3$, where the edge link is a 3-cycle and the edge figure is the triangular 2-cell it bounds. For an edge E , the full stabilizer in the symmetry group acts on the edge figure by the dihedral group D_r , and the orientation-preserving pointwise stabilizer contains a cyclic subgroup C_r acting on the normal 2-plane by rotation through angle $2\pi/r$. The vertex figure is the regular polyhedron of type $\{q, r\}$; equivalently, the local figure at each vertex is $\{q, r\}$. In particular, the valence of each vertex is the number of vertices of the vertex figure. The Schläfli symbols for regular 4-polytopes [4, p. 414] are exhibited in Table 1.

TABLE 1. Vertex and edge figures of the convex regular 4-polytopes

Polytope	Schläfli symbol	Facet	Vertex figure	Edge figure
5-cell	$\{3, 3, 3\}$	$\{3, 3\}$ tetrahedron	$\{3, 3\}$ tetrahedron	$\{3\}$ 3-gon
8-cell	$\{4, 3, 3\}$	$\{4, 3\}$ cube	$\{3, 3\}$ tetrahedron	$\{3\}$ 3-gon
16-cell	$\{3, 3, 4\}$	$\{3, 3\}$ tetrahedron	$\{3, 4\}$ octahedron	$\{4\}$ 4-gon
24-cell	$\{3, 4, 3\}$	$\{3, 4\}$ octahedron	$\{4, 3\}$ cube	$\{3\}$ 3-gon
120-cell	$\{5, 3, 3\}$	$\{5, 3\}$ dodecahedron	$\{3, 3\}$ tetrahedron	$\{3\}$ 3-gon
600-cell	$\{3, 3, 5\}$	$\{3, 3\}$ tetrahedron	$\{3, 5\}$ icosahedron	$\{5\}$ 5-gon

11.3.2. *Rotation subgroup presentations.* We record the following standard presentations for the rotation subgroups of the regular 4-polytopes [17, Eq. (4), p. 1311]. For the regular 4-simplex $\{3, 3, 3\}$, the group presentation for the rotation subgroup is:

$$(11.1) \quad [3, 3, 3]^+ = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^3 = \sigma_2^3 = \sigma_3^3 = (\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2 = 1 \rangle.$$

This group is isomorphic to

$$\langle r, s \mid r^2 = s^3 = (sr)^5 = 1 \rangle \cong A_5$$

under the map $s \mapsto \sigma_1, r \mapsto \sigma_3\sigma_2$.

The regular 8-cell has rotation subgroup:

$$(11.2) \quad [4, 3, 3]^+ = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^4 = \sigma_2^3 = \sigma_3^3 = (\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2 = 1 \rangle.$$

The rotation subgroup of the symmetry group of the regular 24-cell is:

$$(11.3) \quad [3, 4, 3]^+ = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^3 = \sigma_2^4 = \sigma_3^3 = (\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2 = 1 \rangle.$$

For the regular 120-cell, its rotation subgroup is:

$$(11.4) \quad [5, 3, 3]^+ = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^5 = \sigma_2^3 = \sigma_3^3 = (\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2 = 1 \rangle.$$

For the regular 600-cell, its rotation subgroup is:

$$(11.5) \quad [3, 3, 5]^+ = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^3 = \sigma_2^3 = \sigma_3^5 = (\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2 = 1 \rangle.$$

11.4. Even valency of the radial graph. This subsection extracts from Table 1 the even-valency condition needed for the line bundle $\mathcal{I}$ on $S^3 \setminus \Gamma$.

PROPOSITION 11.10 (Even valency of the regular 4-polytopes). *Let $P \subset \mathbb{R}^4$ be the regular 4-polytope with vertex set $\{V_1, \dots, V_n\} \subset \mathbb{R}^4$, and let $\Gamma \subset S^3$ be the radial projection of the 1-skeleton of P . Set $G := \text{Sym}^+(P) \subset SO(4)$. Then each vertex of Γ has even valency.*

PROOF. As the valency of each vertex is the number of vertices of the vertex figure, it is enough to list the vertex figures of the regular 4-polytopes. For the 5-cell and 8-cell, the vertex figure is a tetrahedron; for the 16-cell, it is an octahedron; for the 24-cell, it is a cube; for the 120-cell, it is a tetrahedron; and for the 600-cell, it is an icosahedron. Hence, the corresponding numbers of vertices are 4, 4, 6, 8, 4, 12, respectively. All of these are even. Therefore, each vertex of Γ has even valency. $\square$

11.5. Odd-order pointwise stabilizers of edges and lift of the symmetry action to $\mathcal{I}$. The following proposition is verified case by case in Sections 13–17; see Propositions 13.1, 14.1, 15.1, 16.2, and 17.2 for the model-dependent proofs of the odd-order pointwise stabilizers of edges.

PROPOSITION 11.11 (Odd-order pointwise stabilizers of edges). *Let P be one of the regular 4-polytopes appearing in Theorem 1.2, let $\Gamma \subset S^3$ be the radial projection of its 1-skeleton, and let $G \leq SO(4)$ be the orientation-preserving symmetry group of P . For each edge $E \subset \Gamma$, let $W_E \subset \mathbb{R}^4$ denote the unique 2-plane through the origin such that $E \subset W_E \cap S^3$. Then there exists an odd integer $m > 1$ and a cyclic subgroup $C_m \subset G$ such that C_m fixes E pointwise. Moreover, the induced action of C_m on $W_E^\perp$ is faithful, and a generator of C_m acts on $W_E^\perp$ by rotation through angle $2\pi/m$.*

PROOF. Let P have Schläfli symbol $\{p, q, r\}$. For a regular 4-polytope, exactly r facets meet cyclically around each edge, so the edge figure is a regular r -gon. Hence, the orientation-preserving pointwise stabilizer of an edge acts faithfully on the normal 2-plane as the rotation group of this r -gon, and therefore contains a cyclic subgroup C_r whose generator acts by rotation through angle $2\pi/r$. For the regular 4-polytopes in Theorem 1.2, one has $r = 3$ for the 5-, 8-, 24-, and 120-cell, and $r = 5$ for the 600-cell. Thus r is odd in every case, and the proposition follows with $m = r$. $\square$

Remark. Among the six convex regular 4-polytopes, the only one not discussed here is the 16-cell. In the local models of Section 5, the $\mathbb{Z}/m\mathbb{Z}$ -action lifts to the line bundle $\mathcal{I}$ only when m is odd, and in all of the cases listed here, the corresponding pointwise stabilizer of an edge has odd order. By contrast, the 16-cell has Schläfli symbol $\{3, 3, 4\}$, so its edge figure is a square; equivalently, four tetrahedral facets meet around each edge, and the nontrivial rotations in the cyclic pointwise stabilizer of an edge have orders 2 and 4, not odd order. Therefore, the odd-order lifting criterion from Section 5 fails at an interior edge of the 16-cell, so the construction used in Part II does not apply to it.

PROPOSITION 11.12 (Lift of symmetry to $\mathcal{I}$). *Let P be one of the regular 4-polytopes from Theorem 1.2, let $\Gamma \subset S^3$ be the radial projection of its 1-skeleton, and let $G \leq SO(4)$ be the orientation-preserving symmetry group of P . Then the induced action of G on $S^3 \setminus \Gamma$ lifts to an action on the total space of the line bundle $\mathcal{I} \rightarrow S^3 \setminus \Gamma$ defined in Section 3.*

PROOF. The claim is verified case-by-case in Sections 13–17. In the 5-cell case, Proposition 13.2 proves that the resulting $\mathbb{Z}/2$ -extension splits, and Lemma 12.5 converts the induced G_5 -action on the connected double cover into fiberwise linear bundle automorphisms of the associated line bundle $\mathcal{I}_5 \rightarrow M_5$. The remaining 8-, 24-, 120-, and 600-cell cases are proved in Propositions 14.2, 15.2, 16.3, and 17.4, respectively. Hence, the induced G -action on $S^3 \setminus \Gamma$ lifts to the total space of $\mathcal{I}$ in every case listed in Theorem 1.2. $\square$

12. Monodromy, connected double covers, and fiberwise lifts

This section discusses the covering space and line bundle lemmas used later. Starting from a graph complement $M = S^3 \setminus \Gamma$ and a real line bundle $\mathcal{J} \rightarrow M$ with monodromy character $\mu_{\mathcal{J}} : \pi_1(M) \rightarrow \{\pm 1\}$, this section explains: (i) how monodromy behaves under pullback (Lemma 12.1), (ii) when a homeomorphism of M lifts to the connected double cover associated to $\ker(\mu_{\mathcal{J}})$ (Lemma 12.3), (iii) how such lifts form a central $\mathbb{Z}/2$ -extension of the symmetry group (Lemma 12.4), and (iv) how a choice of lifts forming a group homomorphism induces fiberwise linear automorphisms of the associated real line bundle (Lemma 12.5).

In the polytope applications, Lemma 12.2 is the concrete criterion used to verify the hypothesis of Lemma 12.3: if a symmetry preserves Γ and permutes its edges, then it preserves the monodromy character. Sections 13–17 verify the odd-order cyclic subgroups fixing edges pointwise and the splitting of the lift extension for each model.

12.1. Monodromy and the associated connected double cover.

We begin with the relation between the pullback of real line bundles and the pullback of their monodromy characters. For a self-map, this identifies the condition $f^*\mathcal{J} \simeq \mathcal{J}$ with preservation of the monodromy character, which is the condition used in Lemma 12.2 to lift the map to the associated connected double cover.

LEMMA 12.1 (Pullback monodromy). *Let $f : (X, x_0) \rightarrow (Y, y_0)$ be continuous between path-connected spaces, and let $\mathcal{J} \rightarrow Y$ be a real line bundle. Let $\mu_{\mathcal{J}} : \pi_1(Y, y_0) \rightarrow \{\pm 1\}$ be its monodromy character. Then $\mu_{f^*\mathcal{J}} = \mu_{\mathcal{J}} \circ f_* : \pi_1(X, x_0) \rightarrow \{\pm 1\}$. If in addition, $X = Y$ and X is paracompact, then $f^*\mathcal{J} \simeq \mathcal{J}$ if and only if $\mu_{\mathcal{J}} \circ f_* = \mu_{\mathcal{J}}$.*

PROOF. Fix $x_0 \in X$ and set $y_0 := f(x_0)$. Let $\alpha : (S^1, 1) \rightarrow (X, x_0)$ be a based loop. Then there is a canonical isomorphism of real line bundles over S^1 , $\alpha^*(f^*\mathcal{J}) \cong (f \circ \alpha)^*\mathcal{J}$. Hence the monodromy of $f^*\mathcal{J}$ along α equals the monodromy of $\mathcal{J}$ along $f \circ \alpha$, so $\mu_{f^*\mathcal{J}}([\alpha]) = \mu_{\mathcal{J}}([f \circ \alpha]) = (\mu_{\mathcal{J}} \circ f_*)([\alpha])$. Thus $\mu_{f^*\mathcal{J}} = \mu_{\mathcal{J}} \circ f_*$.

Now assume $X = Y$ and X is paracompact. If $f^*\mathcal{J} \simeq \mathcal{J}$, then isomorphic line bundles have the same monodromy character, so $\mu_{\mathcal{J}} = \mu_{f^*\mathcal{J}} = \mu_{\mathcal{J}} \circ f_*$.

Conversely, suppose that $\mu_{\mathcal{J}} \circ f_* = \mu_{\mathcal{J}}$. By the first part, $\mu_{f^*\mathcal{J}} = \mu_{\mathcal{J}}$. Under the canonical identification $H^1(X; \mathbb{Z}/2) \cong \text{Hom}(\pi_1(X, x_0), \mathbb{Z}/2)$, the first Stiefel–Whitney class $w_1(\mathcal{L})$ of a real line bundle $\mathcal{L}$ corresponds to its monodromy character $\mu_{\mathcal{L}}$ (after identifying $\{\pm 1\} \cong \mathbb{Z}/2$). Therefore, $w_1(f^*\mathcal{J}) = w_1(\mathcal{J})$. Since X is paracompact, the first Stiefel–Whitney class classifies real line bundles on X [16, Ch. 17, Theorem 3.4, p. 250]. Therefore $w_1(f^*\mathcal{J}) = w_1(\mathcal{J})$ implies $f^*\mathcal{J} \simeq \mathcal{J}$. $\square$

LEMMA 12.2 (Edge-permuting symmetries preserve monodromy). *Let $\Gamma \subset S^3$ be a finite graph and set $X := S^3 \setminus \Gamma$. Let $\mathcal{J} \rightarrow X$ be a real line bundle with monodromy character $\mu_{\mathcal{J}} : \pi_1(X) \rightarrow \{\pm 1\}$ such that $\mu_{\mathcal{J}}([m_e]) = -1$ for every edge meridian m_e about an edge $e \subset \Gamma$. Let $f : S^3 \rightarrow S^3$ be a homeomorphism preserving Γ and permuting its edges, and denote by $f : X \rightarrow X$ the induced homeomorphism. Then $\mu_{\mathcal{J}} \circ f_* = \mu_{\mathcal{J}}$.*

PROOF. Since f preserves Γ and permutes its edges, the class $f_*([m_e])$ is conjugate in $\pi_1(X)$ to $[m_{f(e)}]^{\pm 1}$; the sign records whether f preserves the chosen orientation of the meridian. Because every element of $\{\pm 1\}$ equals its own inverse, $\mu_{\mathcal{J}}(\gamma^{-1}) = \mu_{\mathcal{J}}(\gamma)$ for all $\gamma \in \pi_1(X)$. Also, the target $\{\pm 1\}$ is abelian; therefore, $\mu_{\mathcal{J}}$ is invariant under conjugacy. Thus, for each edge e , $(\mu_{\mathcal{J}} \circ f_*)([m_e]) = \mu_{\mathcal{J}}(f_*[m_e]) = \mu_{\mathcal{J}}([m_{f(e)}]^{\pm 1}) = -1 = \mu_{\mathcal{J}}([m_e])$.

Thus $\mu_{\mathcal{J}} \circ f_*$ and $\mu_{\mathcal{J}}$ agree on all edge meridians. Identifying $\{\pm 1\}$ with $\mathbb{Z}/2$, both characters factor through $H_1(X; \mathbb{Z}/2)$. The edge meridians generate $H_1(X; \mathbb{Z}/2)$ by Alexander duality. Hence the two characters agree on $H_1(X; \mathbb{Z}/2)$, and therefore $\mu_{\mathcal{J}} \circ f_* = \mu_{\mathcal{J}}$ on $\pi_1(X)$. $\square$

LEMMA 12.3 (Lifting criterion). *Let X be path-connected and locally path-connected, and let $\mu : \pi_1(X) \rightarrow \{\pm 1\}$ be a nontrivial character. Suppose $p : \tilde{X} \rightarrow X$ is the connected double cover corresponding to $H := \ker(\mu)$, and let τ denote its nontrivial deck involution. For a homeomorphism $f : X \rightarrow X$, the following are equivalent:*

- (1) *f admits a lift $\tilde{f} : \tilde{X} \rightarrow \tilde{X}$ with $p \circ \tilde{f} = f \circ p$;*
- (2) *$f_*(H) = H$;*
- (3) *$\mu \circ f_* = \mu$.*

When these hold, any two lifts differ by composition with τ .

PROOF. Fix $x_0 \in X$ and $\tilde{x}_0 \in p^{-1}(x_0)$, and set $x_1 := f(x_0)$. Choose $\tilde{x}_1 \in p^{-1}(x_1)$ and write, with p_* denoting the homomorphism induced by p on the fundamental groups,

$$H_0 := p_*\pi_1(\tilde{X}, \tilde{x}_0) \leq \pi_1(X, x_0), \quad H_1 := p_*\pi_1(\tilde{X}, \tilde{x}_1) \leq \pi_1(X, x_1).$$

Since $p : \tilde{X} \rightarrow X$ is a connected double cover, the corresponding subgroups have index 2 [13, Proposition 1.32]:

$$[\pi_1(X, x_0) : H_0] = [\pi_1(X, x_1) : H_1] = 2.$$

We first prove (1) $\Leftrightarrow$ (2). If a lift sends $\tilde{x}_0$ to the other point of $p^{-1}(x_1)$, composing it with τ gives a lift sending $\tilde{x}_0$ to the chosen $\tilde{x}_1$. By the covering-space lifting criterion [13, Proposition 1.33], there exists a lift $\tilde{f}$ with $\tilde{f}(\tilde{x}_0) = \tilde{x}_1$ and $p \circ \tilde{f} = f \circ p$ if and only if $f_*(H_0) \subset H_1$, where $f_* : \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$ is induced by f . Indeed, applying the criterion to $f \circ p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_1)$ and the covering $p : (\tilde{X}, \tilde{x}_1) \rightarrow (X, x_1)$ gives the condition

$$(f \circ p)_*\pi_1(\tilde{X}, \tilde{x}_0) \subset p_*\pi_1(\tilde{X}, \tilde{x}_1),$$

which is precisely $f_*(H_0) \subset H_1$.

Assume first that such a lift exists. Then the lifting criterion gives $f_*(H_0) \subset H_1$. Moreover,

$$[\pi_1(X, x_1) : f_*(H_0)] = [\pi_1(X, x_0) : H_0] = 2,$$

because $f_* : \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$ is an isomorphism and hence induces a bijection from the left cosets of H_0 in $\pi_1(X, x_0)$ to the left cosets of $f_*(H_0)$ in $\pi_1(X, x_1)$. Since H_1 also has index 2 in $\pi_1(X, x_1)$, the inclusion $f_*(H_0) \subset H_1$ forces equality: $f_*(H_0) = H_1$. Conversely, $f_*(H_0) = H_1$ implies $f_*(H_0) \subset H_1$, hence the required lift exists. This proves (1) $\Leftrightarrow$ (2).

We next prove $(2) \Leftrightarrow (3)$. If $f_*(H) = H$, then for any loop $[\alpha] \in \pi_1(X, x_0)$ we have $[\alpha] \in H \iff f_*([\alpha]) \in H$, hence

$$\mu(f_*([\alpha])) = \begin{cases} 1 & \text{if } [\alpha] \in H, \\ -1 & \text{if } [\alpha] \notin H, \end{cases} \quad \text{so} \quad \mu \circ f_* = \mu.$$

Conversely, if $\mu \circ f_* = \mu$ and $[\alpha] \in H$, then $\mu([\alpha]) = 1$ implies $\mu(f_*([\alpha])) = 1$, hence $f_*([\alpha]) \in H$; thus $f_*(H) \subset H$. Since $\mu \circ f_* = \mu$ implies $\mu \circ (f^{-1})_* = \mu$, the same argument applied to f^{-1} gives $(f^{-1})_*(H) \subset H$. Applying f_* to both sides gives $H \subset f_*(H)$, hence $f_*(H) = H$.

It remains to prove the assertion about the two possible lifts. If $\tilde{f}_1, \tilde{f}_2$ are two lifts of f , then $\tilde{f}_1(\tilde{x}_0)$ and $\tilde{f}_2(\tilde{x}_0)$ both lie in the fiber $p^{-1}(x_1)$. Since $\text{Deck}(p) = \{\text{id}, \tau\}$ acts transitively on this two-point fiber, there exists $\delta \in \{\text{id}, \tau\}$ such that $\delta(\tilde{f}_1(\tilde{x}_0)) = \tilde{f}_2(\tilde{x}_0)$. Then $\delta \circ \tilde{f}_1$ and $\tilde{f}_2$ are lifts of f that agree at $\tilde{x}_0$, so by the uniqueness of lifts, they are equal. Hence $\tilde{f}_2 = \delta \circ \tilde{f}_1$, i.e. $\tilde{f}_2 = \tilde{f}_1$ or $\tilde{f}_2 = \tau \circ \tilde{f}_1$. $\square$

12.2. Symmetries preserving monodromy and the lift extension. In the polytope cases, the relevant symmetries preserve the graph Γ and permute its edges. Lemma 12.2 shows that such symmetries preserve the monodromy character, and Lemma 12.3 then gives lifts to the associated connected double cover.

Given a subgroup $G \leq \text{Homeo}(M)$, define the group of lifts by

$$\tilde{G} := \{\tilde{g} \in \text{Homeo}(\tilde{M}) \mid \exists g \in G \text{ such that } p \circ \tilde{g} = g \circ p\},$$

where $p : \tilde{M} \rightarrow M$ is the connected double cover associated to $\ker(\mu_{\mathcal{J}})$. The next lemma packages these individual lifts into a short exact sequence and identifies a splitting with a choice of lifts $s(g)$ forming a group homomorphism $s : G \rightarrow \tilde{G}$.

LEMMA 12.4 (Group of lifts and the central extension). *Let $p : \tilde{M} \rightarrow M$ be a connected double cover with deck involution τ , and let $G \leq \text{Homeo}(M)$. Let $\tilde{G}$ denote the set of lifts of elements of G to $\tilde{M}$. Assume that every $g \in G$ admits at least one lift to $\tilde{M}$. Then $\tilde{G}$ is a group under composition, and the projection $\pi : \tilde{G} \rightarrow G$ defined by $\pi(\tilde{g}) = g$ fits into a short exact sequence*

$$1 \longrightarrow \langle \tau \rangle \longrightarrow \tilde{G} \xrightarrow{\pi} G \longrightarrow 1.$$

Moreover, $\langle \tau \rangle \subset Z(\tilde{G})$, so this is a central $\mathbb{Z}/2$ -extension of G . This sequence splits if and only if one can choose, for each $g \in G$, a lift $s(g) \in \tilde{G}$ such that $s(g_1 g_2) = s(g_1) s(g_2)$ for all $g_1, g_2 \in G$.

PROOF. We first show that $\tilde{G}$ is a group under composition. Since $p \circ \text{id}_{\tilde{M}} = \text{id}_M \circ p$, one has $\text{id}_{\tilde{M}} \in \tilde{G}$. If $\tilde{g}_i$ is a lift of g_i for $i = 1, 2$, then

$$p \circ (\tilde{g}_1 \tilde{g}_2) = (p \circ \tilde{g}_1) \circ \tilde{g}_2 = (g_1 \circ p) \circ \tilde{g}_2 = g_1 \circ (p \circ \tilde{g}_2) = (g_1 g_2) \circ p,$$

so $\tilde{g}_1\tilde{g}_2$ is a lift of g_1g_2 . We next show that $\tilde{G}$ is closed under inverses. Let $\tilde{g} \in \tilde{G}$ be a lift of $g \in G$. Since $g^{-1} \in G$, choose a lift $\tilde{h}$ of g^{-1} . Then $\tilde{h}\tilde{g}$ covers id_M , because

$$p \circ (\tilde{h}\tilde{g}) = (p \circ \tilde{h}) \circ \tilde{g} = (g^{-1} \circ p) \circ \tilde{g} = g^{-1} \circ (g \circ p) = \text{id}_M \circ p.$$

After replacing $\tilde{h}$ by $\tau \circ \tilde{h}$ if necessary, we may assume that $(\tilde{h}\tilde{g})(\tilde{x}_0) = \tilde{x}_0$ for a chosen $\tilde{x}_0 \in \tilde{M}$. Thus $\tilde{h}\tilde{g}$ and $\text{id}_{\tilde{M}}$ are lifts of id_M that agree at $\tilde{x}_0$, so uniqueness of lifts gives $\tilde{h}\tilde{g} = \text{id}_{\tilde{M}}$. Similarly, $\tilde{g}\tilde{h}$ and $\text{id}_{\tilde{M}}$ are lifts of id_M that agree at $\tilde{g}(\tilde{x}_0)$, so $\tilde{g}\tilde{h} = \text{id}_{\tilde{M}}$. Hence $\tilde{h} = \tilde{g}^{-1}$, so $\tilde{g}^{-1} \in \tilde{G}$.

We next define the projection to G and check exactness. The projection π is well-defined. Indeed, if $\tilde{g}$ covers both g_1 and g_2 , then $g_1 \circ p = p \circ \tilde{g} = g_2 \circ p$. For any $y \in M$, choose $\tilde{y} \in \tilde{M}$ with $p(\tilde{y}) = y$. Then

$$g_1(y) = g_1(p(\tilde{y})) = g_2(p(\tilde{y})) = g_2(y).$$

Hence $g_1 = g_2$, and $\pi(\tilde{g})$ is unambiguous. Now we check that π is a homomorphism. Suppose that $\pi(\tilde{g}_i) = g_i$ for $i = 1, 2$. Then

$$p \circ (\tilde{g}_1\tilde{g}_2) = (p \circ \tilde{g}_1) \circ \tilde{g}_2 = (g_1 \circ p) \circ \tilde{g}_2 = g_1 \circ (g_2 \circ p) = (g_1g_2) \circ p.$$

Therefore $\pi(\tilde{g}_1\tilde{g}_2) = g_1g_2 = \pi(\tilde{g}_1)\pi(\tilde{g}_2)$.

We now prove exactness. We need to show that the inclusion $\langle \tau \rangle \hookrightarrow \tilde{G}$ is injective, that $\ker(\pi) = \langle \tau \rangle$, and that π is surjective. First, the inclusion $\langle \tau \rangle \hookrightarrow \tilde{G}$ is injective. Indeed, $\text{id}_{\tilde{M}}$ and τ both cover id_M , and $\langle \tau \rangle = \{\text{id}_{\tilde{M}}, \tau\}$. Hence, the natural map $\langle \tau \rangle \hookrightarrow \tilde{G}$ is the inclusion map, so it is injective.

Second, we compute the kernel. Since both $\text{id}_{\tilde{M}}$ and τ cover id_M , one has $\langle \tau \rangle \subset \ker(\pi)$. Conversely, if $\tilde{g} \in \ker(\pi)$, then $\tilde{g}$ covers id_M , so $\tilde{g}$ is a deck transformation of p . Since p is a connected double cover, $\text{Deck}(p) = \{\text{id}_{\tilde{M}}, \tau\}$. Hence $\ker(\pi) = \{\text{id}_{\tilde{M}}, \tau\} = \langle \tau \rangle$. As π is surjective by the hypothesis that every $g \in G$ admits a lift,

$$1 \longrightarrow \langle \tau \rangle \longrightarrow \tilde{G} \xrightarrow{\pi} G \longrightarrow 1$$

is exact. It remains to check that the extension is central, i.e., that $\langle \tau \rangle \subset Z(\tilde{G})$. Let $\tilde{g} \in \tilde{G}$. Then $\tilde{g}\tau\tilde{g}^{-1}$ covers id_M , because

$$p \circ (\tilde{g}\tau\tilde{g}^{-1}) = (g \circ p) \circ \tau \circ \tilde{g}^{-1} = g \circ p \circ \tilde{g}^{-1} = p.$$

Thus $\tilde{g}\tau\tilde{g}^{-1}$ is a deck transformation of p . The deck transformation $\tilde{g}\tau\tilde{g}^{-1}$ is not the identity, otherwise $\tau = \text{id}_{\tilde{M}}$ by multiplying $\tilde{g}$ and $\tilde{g}^{-1}$ on each side. Hence $\tilde{g}\tau\tilde{g}^{-1} = \tau$, or equivalently $\tilde{g}\tau = \tau\tilde{g}$. Therefore $\langle \tau \rangle \subset Z(\tilde{G})$.

Finally, by definition, a splitting is a group homomorphism $s : G \rightarrow \tilde{G}$ with $\pi \circ s = \text{id}_G$. Since $\pi(s(g)) = g$, each $s(g)$ is a lift of g , and since s is a homomorphism it satisfies $s(g_1g_2) = s(g_1)s(g_2)$ for all $g_1, g_2 \in G$. Conversely, any choice of lifts $s(g)$ with $s(g_1g_2) = s(g_1)s(g_2)$ defines a homomorphism $s : G \rightarrow \tilde{G}$ and, since each $s(g)$ is a lift of g , satisfies $\pi \circ s = \text{id}_G$. This proves the final claim. $\square$

12.3. From cover lifts to fiberwise linear bundle automorphisms.

A lift on the connected double cover determines a fiberwise linear automorphism of the associated real line bundle. Replacing a chosen lift by its composition with the deck involution changes the induced bundle map by a sign. Lemma 12.5 connects lifts of G on the double cover $p : \widetilde{M} \rightarrow M$ to fiberwise linear lifts on the associated real line bundle $\mathcal{I} \rightarrow M$ and explains how the deck involution τ produces the sign ambiguity in bundle automorphisms.

LEMMA 12.5 (From lifts of the double cover to fiberwise lifts). *Let $p : \widetilde{M} \rightarrow M$ be a connected double cover with deck involution τ , and let $\mathcal{I} \rightarrow M$ be the associated real line bundle, realized as the quotient*

$$\mathcal{I} \cong (\widetilde{M} \times \mathbb{R}) / ((\tau\tilde{x}, u) \sim (\tilde{x}, -u)).$$

Suppose $g : M \rightarrow M$ is a homeomorphism admitting a lift $\tilde{g} : \widetilde{M} \rightarrow \widetilde{M}$, so that $p \circ \tilde{g} = g \circ p$. Then:

- (1) $\tilde{g}$ commutes with τ .
- (2) The formula $\widehat{g}([\tilde{x}, u]) := [\tilde{g}(\tilde{x}), u]$ defines a well-defined fiberwise linear bundle automorphism $\widehat{g} : \mathcal{I} \rightarrow \mathcal{I}$ covering g . Equivalently, it gives a pullback-bundle isomorphism $\delta_g : g^*\mathcal{I} \rightarrow \mathcal{I}$ by $(\delta_g)_x = (\widehat{g}_x)^{-1} : \mathcal{I}_{g(x)} \rightarrow \mathcal{I}_x$.
- (3) Replacing $\tilde{g}$ by the other lift $\tau \circ \tilde{g}$ replaces $\widehat{g}$ by $-\widehat{g}$.

PROOF. We first show that $\tilde{g}$ is a homeomorphism. This makes the conjugate $\tilde{g}\tau\tilde{g}^{-1}$ well-defined; it also allows the construction applied to $\tilde{g}^{-1}$ to give an inverse to the induced bundle map, so that the resulting fiberwise linear map is a bundle automorphism. Let $\mu_{\mathcal{I}} : \pi_1(M) \rightarrow \{\pm 1\}$ denote the monodromy character of $\mathcal{I}$. Equivalently, $p : \widetilde{M} \rightarrow M$ is the connected double cover corresponding to $\ker(\mu_{\mathcal{I}})$. Since $\tilde{g}$ is a lift of g , Lemma 12.3 gives $\mu_{\mathcal{I}} \circ g_* = \mu_{\mathcal{I}}$. Let $h = g^{-1}$. Since $h_* = g_*^{-1}$ and $\mu_{\mathcal{I}} \circ g_* = \mu_{\mathcal{I}}$, composing on the right by g_*^{-1} gives $\mu_{\mathcal{I}} \circ h_* = \mu_{\mathcal{I}}$. Hence, by Lemma 12.3, h admits a lift $\tilde{h} : \widetilde{M} \rightarrow \widetilde{M}$. Choose $\tilde{x}_0 \in \widetilde{M}$, and replace $\tilde{h}$ by $\tau \circ \tilde{h}$ if necessary so that $\tilde{h}(\tilde{g}(\tilde{x}_0)) = \tilde{x}_0$. Then $\tilde{h} \circ \tilde{g}$ is a lift of id_M , since

$$p \circ (\tilde{h} \circ \tilde{g}) = (p \circ \tilde{h}) \circ \tilde{g} = (h \circ p) \circ \tilde{g} = h \circ (g \circ p) = \text{id}_M \circ p.$$

It agrees with $\text{id}_{\widetilde{M}}$ at $\tilde{x}_0$, so by uniqueness of lifts, $\tilde{h} \circ \tilde{g} = \text{id}_{\widetilde{M}}$. Similarly,

$$p \circ (\tilde{g} \circ \tilde{h}) = (p \circ \tilde{g}) \circ \tilde{h} = (g \circ p) \circ \tilde{h} = g \circ (h \circ p) = p,$$

and $(\tilde{g} \circ \tilde{h})(\tilde{g}(\tilde{x}_0)) = \tilde{g}(\tilde{x}_0)$, so again by the uniqueness of lifts, $\tilde{g} \circ \tilde{h} = \text{id}_{\widetilde{M}}$. Thus $\tilde{h} = \tilde{g}^{-1}$, and in particular $\tilde{g}$ is a homeomorphism.

(1) Apply Lemma 12.4 to the cyclic subgroup $\langle g \rangle \leq \text{Homeo}(M)$. Its hypothesis is satisfied because $\tilde{g}^n$ is a lift of g^n for every $n \in \mathbb{Z}$. Since $\tilde{g}$ belongs to the corresponding lift group, the centrality conclusion of Lemma 12.4 gives $\tilde{g}\tau = \tau\tilde{g}$.

(2) If $(\tau\tilde{x}, u) \sim (\tilde{x}, -u)$, then using (1),

$$[\tilde{g}(\tau\tilde{x}), u] = [\tau\tilde{g}(\tilde{x}), u] = [\tilde{g}(\tilde{x}), -u],$$

so $\widehat{g}$ is well-defined on the quotient. It is fiberwise linear because

$$\begin{aligned}\widehat{g}([\tilde{x}, u] + [\tilde{x}, v]) &= [\tilde{g}(\tilde{x}), u + v] = \widehat{g}([\tilde{x}, u]) + \widehat{g}([\tilde{x}, v]), \\ \widehat{g}(\lambda[\tilde{x}, u]) &= [\tilde{g}(\tilde{x}), \lambda u] = \lambda \widehat{g}([\tilde{x}, u]).\end{aligned}$$

Moreover, if $\pi : \mathcal{J} \rightarrow M$ denotes the bundle projection, then

$$\pi(\widehat{g}([\tilde{x}, u])) = p(\tilde{g}(\tilde{x})) = g(p(\tilde{x})) = g(\pi([\tilde{x}, u])),$$

so $\widehat{g}$ covers g .

Since $\tilde{g}^{-1}$ is a lift of g^{-1} , the same construction applied to $\tilde{g}^{-1}$ gives an inverse bundle map. Hence $\widehat{g}$ is a bundle automorphism. Equivalently, since $\widehat{g}$ is an automorphism covering g , its inverse on fibers gives a pullback-bundle isomorphism

$$(\delta_g)_x = (\widehat{g}_x)^{-1} : \mathcal{J}_{g(x)} \rightarrow \mathcal{J}_x, \quad \delta_g : g^* \mathcal{J} \rightarrow \mathcal{J}.$$

This is the fiberwise direction used in Section 5.

(3) If $\tilde{g}' = \tau \circ \tilde{g}$, then $\widehat{g}'([\tilde{x}, u]) = [\tau\tilde{g}(\tilde{x}), u] = [\tilde{g}(\tilde{x}), -u] = -\widehat{g}([\tilde{x}, u])$. $\square$

REMARK 12.6. *Part (2) may also be viewed through the functoriality of associated bundles. Since $\mathcal{J} \cong \widetilde{M} \times_{\mathbb{Z}/2} \mathbb{R}_{\text{sgn}}$, a lift $\tilde{g} : \widetilde{M} \rightarrow \widetilde{M}$ commuting with the deck involution is a principal $\mathbb{Z}/2$ -bundle automorphism, so Proposition 6.2 of Husemoller induces the associated fiberwise linear bundle automorphism of $\mathcal{J}$ [16]. If the chosen lifts come from a splitting $s : G \rightarrow \widetilde{G}$, then functoriality gives $\widehat{s(g_1 g_2)} = \widehat{s(g_1)} \widehat{s(g_2)}$, so the induced bundle automorphisms form a G -action on $\mathcal{J}$.*

13. Order-3 isometries on regular 5-cell

Let $P_5 \subset \mathbb{R}^4$ be the regular 5-cell, let $\Gamma_5 \subset S^3$ be the radial projection of its 1-skeleton, set $M_5 := S^3 \setminus \Gamma_5$, and let $G_5 := \text{Sym}^+(P_5) \cong A_5$. By Section 12, the remaining tasks are to identify the odd-order pointwise stabilizers of edges needed for Proposition 11.11 and to show that the associated short exact sequence $1 \rightarrow \langle \tau_5 \rangle \rightarrow \widetilde{G}_5 \rightarrow G_5 \rightarrow 1$ splits.

13.1. The 5-cell model. A 5-cell (regular 4-simplex) is the convex hull of five points in $\mathbb{R}^4$ with all pairwise distances equal; see e.g. [5]. It has Schläfli symbol $\{3, 3, 3\}$, so its facets and vertex figures are regular tetrahedra. In this subsection, we fix explicit vertices $V_1^{(5)}, \dots, V_5^{(5)} \in S^3 \subset \mathbb{R}^4 \simeq \mathbb{H}$ adapted to the quaternionic order-3 rotation Ad_q , and we record the induced permutation of vertices and tetrahedral facets under Ad_q .

The symmetry group of the 5-cell is isomorphic to S_5 , acting by permuting the vertices, and its rotational subgroup is A_5 . Let

$$F := \text{span}_{\mathbb{R}}\{1, i + j + k\} \subset \mathbb{H}$$

denote the fixed 2-plane of Ad_q in $\mathbb{R}^4 \simeq \mathbb{H}$. We choose $V_1^{(5)}, V_2^{(5)} \in F \cap S^3$ and $V_3^{(5)}, V_4^{(5)}, V_5^{(5)}$ so that Ad_q cyclically permutes $\{V_3^{(5)}, V_4^{(5)}, V_5^{(5)}\}$.

The five vertices

$$\begin{aligned} V_1^{(5)} &= (1, 0, 0, 0), & V_2^{(5)} &= \left(-\frac{1}{4}, s, s, s\right), & V_3^{(5)} &= \left(-\frac{1}{4}, s, -s, -s\right), \\ V_4^{(5)} &= \left(-\frac{1}{4}, -s, s, -s\right), & V_5^{(5)} &= \left(-\frac{1}{4}, -s, -s, s\right) \end{aligned}$$

where $s := \frac{\sqrt{5}}{4}$. One checks that $V_i^{(5)} \in S^3$ and that $\|V_i^{(5)} - V_j^{(5)}\|^2 = 5/2$ for $i \neq j$; hence the $V_i^{(5)}$ form a regular 4-simplex whose centroid is the origin. Thus P_5 has 5 vertices, 10 edges, and 5 tetrahedral 3-cells.

Using the identification $\mathbb{R}^4 \cong \mathbb{H}$ from Section 11.1, consider

$$q = \frac{1}{2}(1 + i + j + k) \in S^3, \quad \text{Ad}_q(x) = q x (q)^{-1}.$$

On $\mathfrak{S}\mathbb{H} \cong \mathbb{R}^3$ it is a 120° rotation about the axis $\text{Span}\{i + j + k\}$, and in $\mathbb{R}^4$ its fixed-point set is the plane $\text{Span}\{1, i + j + k\}$. This choice of q will be used in Sections 13–16.

The images of vertices under Ad_q are as follows:

$$\text{Ad}_q : V_3^{(5)} \longrightarrow V_4^{(5)} \longrightarrow V_5^{(5)} \longrightarrow V_3^{(5)}, \quad V_1^{(5)}, V_2^{(5)} \text{ fixed.}$$

For any index i , let

$$T_i^{(5)} := \text{Hull}\{\{V_1^{(5)}, V_2^{(5)}, V_3^{(5)}, V_4^{(5)}, V_5^{(5)}\} \setminus \{V_i^{(5)}\}\}$$

be the regular tetrahedral facet opposite $V_i^{(5)}$. Every facet has edge length $\sqrt{5/2}$ and all ten triangular faces are congruent equilateral triangles. Any two distinct facets share exactly one triangular face, so the facet adjacency graph, whose vertices are the facets $T_i^{(5)}$ and whose edges record shared triangular faces, is the complete graph K_5 .

By computing the quaternion conjugation, Ad_q fixes the 2-plane $F = \text{span}_{\mathbb{R}}\{1, i + j + k\}$ pointwise and rotates $(F)^\perp$ by angle $2\pi/3$. Hence $\text{Fix}(\text{Ad}_q|_{S^3}) = F \cap S^3$, so Ad_q fixes $V_1^{(5)}, V_2^{(5)}$ and cyclically permutes $V_3^{(5)}, V_4^{(5)}, V_5^{(5)}$. Consequently, the induced permutation on facets is $(T_1^{(5)})(T_2^{(5)})(T_3^{(5)} T_4^{(5)} T_5^{(5)})$, i.e. $\text{Ad}_q : T_3^{(5)} \mapsto T_4^{(5)} \mapsto T_5^{(5)} \mapsto T_3^{(5)}$.

13.1.1. *Lifted facet decomposition and the double cover.* Let $\Gamma_5 \subset S^3$ be the radial projection of the 1-skeleton of P_5 , and set $M_5 := S^3 \setminus \Gamma_5$. Here $\pi : \mathbb{R}^4 \setminus \{0\} \rightarrow S^3$ denotes the radial projection $x \mapsto x/|x|$. Let $\mu_5 : \pi_1(M_5) \rightarrow \{\pm 1\}$ denote the monodromy character determined by $\mu_5([m_e]) = -1$ on edge meridians. Let

$$\mathcal{T}_i^{(5)} := \pi\left(T_i^{(5)}\right) \subset S^3, \quad \mathring{\mathcal{T}}_i^{(5)} := \mathcal{T}_i^{(5)} \setminus \Gamma_5 \subset M_5.$$

Let $p_5 : \widetilde{M}_5 \rightarrow M_5$ be the associated connected double cover corresponding to the kernel of the monodromy character μ_5 and let τ_5 denote its nontrivial deck involution. Therefore, the preimage $p_5^{-1}\left(\mathring{\mathcal{T}}_i^{(5)}\right)$ is the disjoint union of

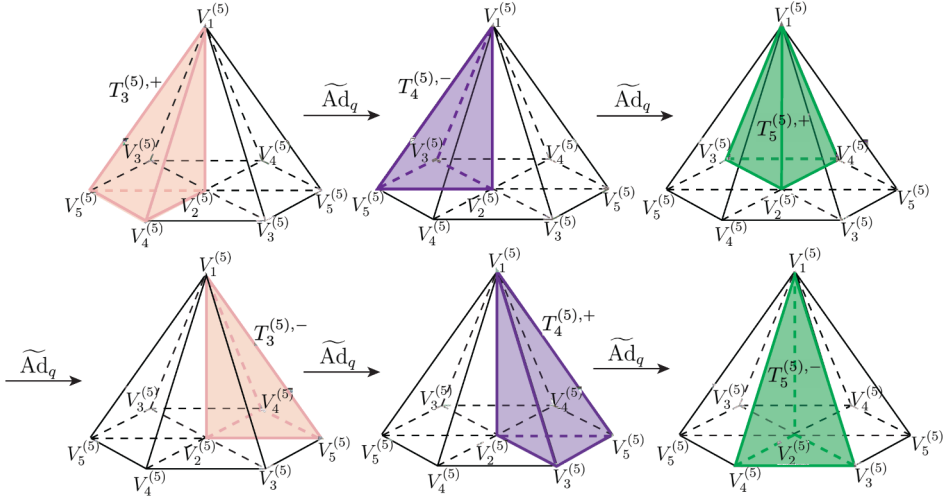


FIGURE 4. Lifted facet dynamics for the order-6 lift $\widetilde{\text{Ad}}_q^{\text{perm}}$ over the edge $[V_1^{(5)}, V_2^{(5)}]$. Here $T_i^{(5),\pm}$ denotes the two lifts of the facet $T_i^{(5)}$, and τ_5 is the deck involution. Reading left to right across the first row and then the second row gives $T_3^{(5),+} \rightarrow T_4^{(5),-} \rightarrow T_5^{(5),+} \rightarrow T_3^{(5),-} \rightarrow T_4^{(5),+} \rightarrow T_5^{(5),-}$, with $(\widetilde{\text{Ad}}_q^{\text{perm}})^3 = \tau_5$.

two copies of $\mathring{\mathcal{T}}_i^{(5)}$; after fixing one choice of sheet we write

$$p_5^{-1} \left(\mathring{\mathcal{T}}_i^{(5)} \right) = \mathcal{T}_i^{(5),+} \sqcup \mathcal{T}_i^{(5),-}, \quad \tau_5 \left(\mathcal{T}_i^{(5),\pm} \right) = \mathcal{T}_i^{(5),\mp}.$$

For $i \neq j$ let $F_{ij}^{(5)} := \mathcal{T}_i^{(5)} \cap \mathcal{T}_j^{(5)}$ denote the common triangular face, and set $\mathring{F}_{ij}^{(5)} := F_{ij}^{(5)} \setminus \Gamma_5 \subset M_5$. Then $p_5^{-1} \left(\mathring{F}_{ij}^{(5)} \right) = F_{ij}^{(5),+} \sqcup F_{ij}^{(5),-}$ with $\tau_5 \left(F_{ij}^{(5),\pm} \right) = F_{ij}^{(5),\mp}$.

Since Ad_q preserves Γ_5 and permutes its edges, Lemma 12.2 gives $\mu_5 \circ (\text{Ad}_q)_* = \mu_5$. Hence, by Lemma 12.3, the symmetry $\text{Ad}_q \in G_5$ admits a lift to $\widetilde{M}_5$, and any two such lifts differ by composition with τ_5 . Since Ad_q has order 3 on M_5 , one lift has order 3, and the other has order 6. For the lifted-facet illustration in Fig. 4, choose the order-6 lift $\widetilde{\text{Ad}}_q^{\text{perm}}$ characterized by $(\widetilde{\text{Ad}}_q^{\text{perm}})^3 = \tau_5$. Its action on the lifted facets is the permutation

$$\begin{aligned} \widetilde{\text{Ad}}_q^{\text{perm}} : & \left(\mathcal{T}_1^{(5),+} \mathcal{T}_1^{(5),-} \right) \left(\mathcal{T}_2^{(5),+} \mathcal{T}_2^{(5),-} \right) \\ & \left(\mathcal{T}_3^{(5),+} \mathcal{T}_4^{(5),-} \mathcal{T}_5^{(5),+} \mathcal{T}_3^{(5),-} \mathcal{T}_4^{(5),+} \mathcal{T}_5^{(5),-} \right), \end{aligned}$$

as illustrated in Fig. 4. The order-3 lift from the splitting is chosen later in Proposition 13.2.

13.2. Invariance of μ_5 and lifting to $\widetilde{M}_5$. This subsection shows that for 5-cells, the natural $G_5 = \text{Sym}^+(P_5) \cong A_5$ -action on M_5 preserves the monodromy character $\mu_5 : \pi_1(M_5) \rightarrow \{\pm 1\}$, and hence lifts to the connected double cover $p_5 : \widetilde{M}_5 \rightarrow M_5$. Let $\mathcal{I}_5 \rightarrow M_5$ denote the real line bundle associated to $p_5 : \widetilde{M}_5 \rightarrow M_5$. Proposition 13.1 supplies, for each edge of P_5 , an element of odd order in G_5 fixing that edge pointwise, to fulfill the 5-cell case of Proposition 11.11. By Proposition 11.10, every vertex of Γ_5 has valency 4, so the assignment $\mu_5([m_e]) = -1$ for each edge meridian m_e is well-defined. Since every $g \in G_5$ preserves Γ_5 and permutes its edges, Lemma 12.2 implies that $\mu_5 \circ g_* = \mu_5$ for every $g \in G_5$. By Lemma 12.3, this is equivalent to saying that each $g \in G_5$ lifts to the connected double cover $p_5 : \widetilde{M}_5 \rightarrow M_5$, uniquely up to the deck involution τ_5 . Lemma 12.4 packages these individual lifts into the short exact sequence

$$1 \longrightarrow \langle \tau_5 \rangle \longrightarrow \widetilde{G}_5 \xrightarrow{\pi_5} G_5 \longrightarrow 1.$$

Proposition 13.2 proves that this extension splits. Equivalently, there is a homomorphism $\sigma : G_5 \rightarrow \widetilde{G}_5$ with $\pi_5 \circ \sigma = \text{id}_{G_5}$. Lemma 12.5 then converts the chosen lifts $\sigma(g)$ into fiberwise linear bundle automorphisms of the associated line bundle $\mathcal{I}_5 \rightarrow M_5$.

PROPOSITION 13.1 (Order-3 symmetries of the 5-cell). *Let $P_5 \subset \mathbb{R}^4$ be the regular 5-cell with vertex set $\{V_1^{(5)}, \dots, V_5^{(5)}\} \subset \mathbb{R}^4$, and let $\Gamma_5 \subset S^3$ be the radial projection of the 1-skeleton of P_5 . Set $G_5 := \text{Sym}^+(P_5) \cong A_5 \subset SO(4)$. Let $E_0^{(5)} := [V_1^{(5)}, V_2^{(5)}]$. For any edge $E^{(5)} = [V_i^{(5)}, V_j^{(5)}]$ of P_5 there exists $h \in G_5$ with $h(E_0^{(5)}) = E^{(5)}$, and the two nontrivial order-3 elements of G_5 fixing $E^{(5)}$ pointwise are $h \text{Ad}_q h^{-1}$ and its inverse. Each restricts to $W_{E^{(5)}}^\perp$, where $W_{E^{(5)}} := \text{Span}_{\mathbb{R}}\{V_i^{(5)}, V_j^{(5)}\}$, to a rotation by $\pm 2\pi/3$. In particular, G_5 contains exactly 20 elements of order 3.*

PROOF. By §13.1, the map $\text{Ad}_q \in G_5 \subset SO(4)$ has order 3 and fixes pointwise the 2-plane $F = \text{Fix}(\text{Ad}_q)$, rotating $F^\perp$ by $2\pi/3$. Moreover, $V_1^{(5)}$ and $V_2^{(5)}$ lie in F , so Ad_q fixes the edge $E_0^{(5)}$ pointwise. Since $V_1^{(5)}$ and $V_2^{(5)}$ are linearly independent and F is a 2-plane, we have

$$F = \text{span}_{\mathbb{R}}\{V_1^{(5)}, V_2^{(5)}\} = W_{E_0^{(5)}}.$$

Therefore Ad_q restricting to $W_{E_0^{(5)}}^\perp$ is a rotation by $2\pi/3$. Since $G_5 \cong A_5$ acts transitively on the 2-element subsets of $\{1, \dots, 5\}$, for any edge $E^{(5)}$ there exists $h \in G_5$ with $h(E_0^{(5)}) = E^{(5)}$. Then $h \text{Ad}_q h^{-1}$ fixes $E^{(5)}$ pointwise and fixes $W_{E^{(5)}} = h(W_{E_0^{(5)}})$ pointwise. Hence its restriction to $W_{E^{(5)}}^\perp$ is a rotation by $\pm 2\pi/3$. Any order-3 symmetry fixing both endpoints of $E^{(5)}$ must act as a 3-cycle on the remaining three vertices; there are exactly two

such 3-cycles. As P_5 has $\binom{5}{2} = 10$ edges, G_5 contains exactly 20 elements of order 3. $\square$

13.3. From lifts of the double cover to fiberwise lifts. Given $g^*\mathcal{I} \simeq \mathcal{I}$, Lemma 12.1 gives $\mu \circ g_* = \mu$, thus μ is G -invariant. Therefore, from Lemma 12.3, one knows that every $g \in G_5$ admits a lift. We then define the *group of lifts*

$$\tilde{G}_5 := \{\tilde{g} \in \text{Homeo}(\tilde{M}_5) \mid \exists g \in G_5 \text{ such that } p_5 \circ \tilde{g} = g \circ p_5\}.$$

Lemma 12.5 shows that for each $g \in G_5$, a choice of lift $\tilde{g} : \tilde{M}_5 \rightarrow \tilde{M}_5$ induces a fiberwise linear bundle automorphism $\hat{g} : \mathcal{I}_5 \rightarrow \mathcal{I}_5$, and replacing $\tilde{g}$ by $\tau_5 \circ \tilde{g}$ replaces $\hat{g}$ by $-\hat{g}$. Thus a splitting of the extension in Lemma 12.4 gives a choice of lifts whose induced bundle maps $\hat{g}$ satisfy $\widehat{gh} = \hat{g} \circ \hat{h}$ for all $g, h \in G_5$. Proposition 13.2 constructs such a splitting when $G_5 = \text{Sym}^+(P_5) \cong A_5$.

PROPOSITION 13.2 (Splitting for the 5-cell group). *The G_5 -action on M_5 lifts to a fiberwise linear action by bundle automorphisms $\hat{g} : \mathcal{I}_5 \rightarrow \mathcal{I}_5$ covering g , such that $\widehat{gh} = \hat{g} \circ \hat{h}$ for all $g, h \in G_5$.*

PROOF. Let $H_5 := \ker(\mu_5)$. By Lemma 12.2, for every $g \in G_5$ we have $\mu_5 \circ g_* = \mu_5$. Since $p_5 : \tilde{M}_5 \rightarrow M_5$ is the connected double cover corresponding to H_5 , Lemma 12.3 implies that $g_*(H_5) = H_5$ and that every $g \in G_5$ admits a lift to $\tilde{M}_5$, unique up to composition with the deck involution τ_5 .

Let $\pi_5 : \tilde{G}_5 \rightarrow G_5$ be given by $\pi_5(\tilde{g}) = g$. By Lemma 12.4, this fits into a short exact sequence

$$1 \longrightarrow \langle \tau_5 \rangle \longrightarrow \tilde{G}_5 \xrightarrow{\pi_5} G_5 \longrightarrow 1,$$

so it suffices to construct a splitting $\sigma : G_5 \rightarrow \tilde{G}_5$.

According to Eq. (11.1), $G_5 \cong A_5$ has the presentation

$$A_5 = \langle r, s \mid r^2 = 1, s^3 = 1, (rs)^5 = 1 \rangle.$$

Step 1: choose a lift of s of order 3. By Lemma 12.3, s admits a lift $\tilde{s}$ to $\tilde{M}_5$. Since $s^3 = \text{id}$, the element $\tilde{s}^3$ is a deck transformation, so $\tilde{s}^3 \in \{\text{id}, \tau_5\}$. If $\tilde{s}^3 = \tau_5$, replace $\tilde{s}$ by $\tau_5 \tilde{s}$. By Lemma 12.5(1), every lift commutes with τ_5 , hence $(\tau_5 \tilde{s})^3 = \tau_5^3 \tilde{s}^3 = \tau_5 \cdot \tau_5 = \text{id}$. Thus, we may choose a lift $\tilde{s}$ such that $\tilde{s}^3 = \text{id}$.

Step 2: choose a lift of r of order 2. As an involution in A_5 , r is a double transposition on the five vertices, so it fixes exactly one vertex of P_5 . Since $r^2 = \text{id}$, the minimal polynomial of r divides $x^2 - 1 = (x - 1)(x + 1)$, so r is diagonalizable with eigenvalues in $\{\pm 1\}$. As $r \in SO(4)$, this implies $\det(r) = 1$, so the multiplicity of the eigenvalue -1 is even. Since $r \neq \pm \text{id}$, the only possibility is that both $+1$ and -1 occur with multiplicity 2. Hence $\ker(r - \text{id})$ is a 2-plane in $\mathbb{R}^4$, and $\text{Fix}(r|_{S^3}) = \ker(r - \text{id}) \cap S^3$ is a great circle. This circle cannot be contained in Γ_5 . Indeed, if it were contained in the finite graph Γ_5 , then it would contain the radial image of some edge E of

P_5 , and r would fix that edge pointwise. But the 5-cell has Schläfli symbol $\{3, 3, 3\}$, so three tetrahedral facets meet around each edge and the edge figure is a triangle. Hence, the orientation-preserving pointwise stabilizer of E acts faithfully as a subgroup of the rotation group of this triangle, namely C_3 . Thus, every nontrivial element fixing E pointwise has order 3, whereas r has order 2, a contradiction. Hence $\text{Fix}(r|_{S^3}) \cap M_5 \neq \emptyset$. Choose $x_r \in \text{Fix}(r|_{S^3}) \setminus \Gamma_5$, a lift $\tilde{x}_r \in p_5^{-1}(x_r)$, and let $\tilde{r}$ be the unique lift of r such that $\tilde{r}(\tilde{x}_r) = \tilde{x}_r$. Then $\tilde{r}^2$ is a deck transformation, since it covers $r^2 = \text{id}$. Because $\tilde{r}^2$ fixes $\tilde{x}_r$ and τ_5 acts freely, it follows that $\tilde{r}^2 = \text{id}$. The other lift $\tau_5\tilde{r}$ also has square id , thus both lifts of r have order 2, and Step 3 chooses between them to enforce the relation $(rs)^5 = 1$ upstairs.

Step 3: choose between the two order-2 lifts of r to enforce the relation $(rs)^5 = 1$ upstairs. The lift $\tilde{s}$ is fixed from Step 1. The lift $\tilde{r}$ from Step 2 is still provisional: in this step, we choose between $\tilde{r}$ and $\tau_5\tilde{r}$. The product $\tilde{r}\tilde{s}$ is a lift of rs , since $p_5 \circ (\tilde{r}\tilde{s}) = (rs) \circ p_5$. Iterating gives $p_5 \circ (\tilde{r}\tilde{s})^5 = (rs)^5 \circ p_5 = p_5$. Hence $(\tilde{r}\tilde{s})^5 \in \{\text{id}, \tau_5\}$. The desired relation for the splitting is $(\tilde{r}\tilde{s})^5 = \text{id}$. If instead $(\tilde{r}\tilde{s})^5 = \tau_5$, replace $\tilde{r}$ by the other order-2 lift $\tau_5\tilde{r}$. Since τ_5 is central in $\tilde{G}_5$, $((\tau_5\tilde{r})\tilde{s})^5 = \tau_5^5(\tilde{r}\tilde{s})^5 = \tau_5 \cdot \tau_5 = \text{id}$. After this possible replacement, we continue to denote the chosen lift by $\tilde{r}$. Hence, the final choices satisfy the defining relations

$$\tilde{r}^2 = \text{id}, \quad \tilde{s}^3 = \text{id}, \quad (\tilde{r}\tilde{s})^5 = \text{id}.$$

Therefore the assignment $\sigma(r) = \tilde{r}$, $\sigma(s) = \tilde{s}$ extends to a homomorphism $\sigma : G_5 \rightarrow \tilde{G}_5$ with $\pi_5 \circ \sigma = \text{id}_{G_5}$.

For each $g \in G_5$, let $\hat{g} : \mathcal{I}_5 \rightarrow \mathcal{I}_5$ be the bundle map induced by the lift $\sigma(g)$ via Lemma 12.5. Then $\hat{g}h = \hat{g} \circ \hat{h}$ follows from $\sigma(gh) = \sigma(g)\sigma(h)$. $\square$

REMARK 13.3 (The two lifts of Ad_q in the 5-cell case). *Let $\sigma : G_5 \rightarrow \tilde{G}_5$ denote the splitting constructed in the proof of Proposition 13.2. Then $\sigma(\text{Ad}_q)$ is a lift of Ad_q of order 3. The order-6 lift used in §13.1.1 to describe the lifted facets is the other lift, namely*

$$\widetilde{\text{Ad}}_q^{\text{perm}} = \tau_5 \circ \sigma(\text{Ad}_q),$$

and hence $(\widetilde{\text{Ad}}_q^{\text{perm}})^3 = \tau_5$. Thus Proposition 13.2 uses the order-3 lift from the splitting, whereas §13.1.1 exhibits the order-6 lift.

14. Order-3 isometries on regular 8-cell

Let $P_8 := [-1, 1]^4 \subset \mathbb{R}^4$ denote the regular 8-cell, let $\Gamma_8 \subset S^3$ be the radial projection of its 1-skeleton, and set $M_8 := S^3 \setminus \Gamma_8$. Let $G_8 := \text{Sym}^+(P_8)$ denote the orientation-preserving symmetry group of P_8 . The goal of this section is to prove the 8-cell cases of Propositions 11.11 and 11.12. Using the explicit model of P_8 , Proposition 14.1 shows that for each edge $E^{(8)} \subset P_8$, there are two nontrivial order-3 elements of G_8 fixing $E^{(8)}$ pointwise and acting on $W_{E^{(8)}}^\perp$ by rotation through angles $\pm 2\pi/3$; this gives the 8-cell case

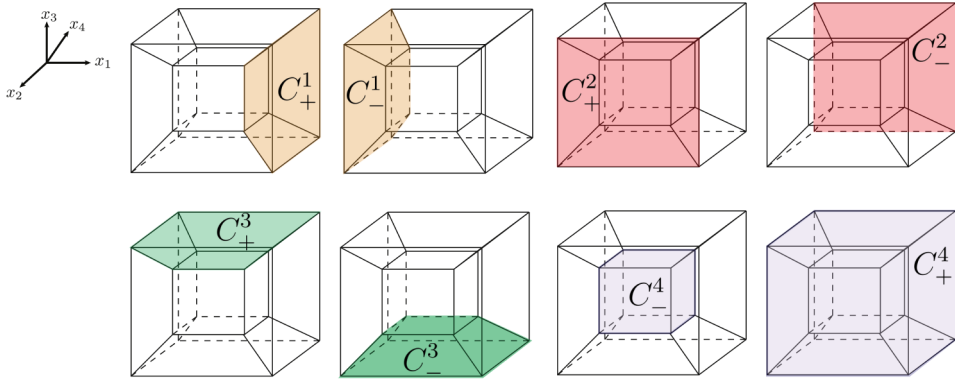


FIGURE 5. The eight Euclidean cubic facets of the 8-cell $P_8 = [-1, 1]^4$: $C^1_+ = (+1, r, s, t)$, $C^1_- = (-1, r, s, t)$, $C^2_+ = (r, +1, s, t)$, $C^2_- = (r, -1, s, t)$, $C^3_+ = (r, s, +1, t)$, $C^3_- = (r, s, -1, t)$, $C^4_+ = (r, s, t, +1)$, $C^4_- = (r, s, t, -1)$, with $r, s, t \in [-1, 1]$.

of Proposition 11.11. Proposition 11.10 shows that every vertex of Γ_8 has valency 4, so the assignment $\mu_8([m_e]) = -1$ on edge meridians is well-defined. Proposition 14.2 proves that this extension splits and therefore the G_8 -action lifts through a homomorphism $s : G_8 \rightarrow \tilde{G}_8$.

14.1. The 8-cell model. The regular 8-cell is the 4-dimensional cube; see e.g. [5]. It has Schläfli symbol $\{4, 3, 3\}$, so its facets are Euclidean cubes and its vertex figures are regular tetrahedra. P_8 has 16 vertices, 32 edges, and 8 cubic 3-cells. Its edge length is 2, and every vertex has distance 2 from the origin. The vertices of P_8 are the 16 sign vectors $(\pm 1, \pm 1, \pm 1, \pm 1)$; we enumerate them lexicographically as $V_1^{(8)}, \dots, V_{16}^{(8)}$, so that

$$\begin{aligned} V_1^{(8)} &= (+1, +1, +1, +1), & V_8^{(8)} &= (+1, -1, -1, -1), \\ V_9^{(8)} &= (-1, +1, +1, +1), & V_{16}^{(8)} &= (-1, -1, -1, -1). \end{aligned}$$

For $i \in \{1, 2, 3, 4\}$ and $\varepsilon \in \{\pm 1\}$, let

$$C_\varepsilon^i := \{(x_1, x_2, x_3, x_4) \in \partial P_8 : x_i = \varepsilon\}$$

denote the cubic facet obtained by fixing the i -th coordinate. Two distinct facets C_ε^i and C_δ^j meet along a square face if and only if $i \neq j$; the opposite pair C_+^i and C_-^i are disjoint. Fig. 5 records this labeling of the eight Euclidean cubic facets.

In coordinates, Ad_q acts by

$$(14.1) \quad \text{Ad}_q(x_1, x_2, x_3, x_4) = (x_1, x_4, x_2, x_3).$$

Since Ad_q fixes the first coordinate and cyclically permutes the last three, the coordinate formula (14.1) shows that Ad_q has order 3 and preserves P_8 , hence permutes the vertices, edges, and facets of P_8 . By quaternion conjugation,

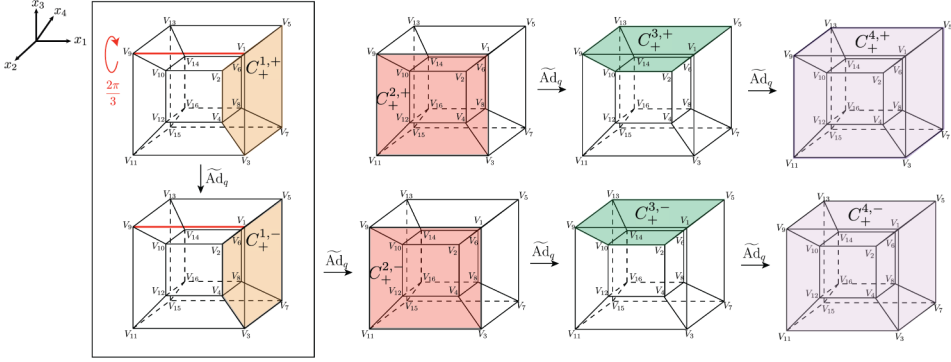


FIGURE 6. Lifted facet dynamics for the order-3 lift $\widetilde{\text{Ad}}_q$.

We write $V_i := V_i^{(8)}$. The figure displays the two cycles over the base facets C_+^2, C_+^3 , and C_+^4 . The two cycles are: $C_+^{2,+} \rightarrow C_+^{3,+} \rightarrow C_+^{4,+}$ (first row), and $C_+^{2,-} \rightarrow C_+^{3,-} \rightarrow C_+^{4,-}$ (second row). The analogous cycles over C_-^2, C_-^3, C_-^4 are not shown. The lifted facets over C_+^1 and C_-^1 are fixed by this order-3 lift.

Ad_q fixes the 2-plane $F = \text{Span}_{\mathbb{R}}\{1, i + j + k\}$ pointwise and rotates $F^\perp$ by angle $2\pi/3$. Hence Ad_q fixes pointwise the two edges $E_+^{(8)} = [V_1^{(8)}, V_9^{(8)}]$ and $E_-^{(8)} = [V_8^{(8)}, V_{16}^{(8)}]$, preserves the two facets $C_\pm^1$, and cyclically permutes $(C_+^2, C_+^3, C_+^4)(C_-^2, C_-^3, C_-^4)$. This facet motion is illustrated schematically in Fig. 6: the two facets $C_\pm^1$ are setwise invariant, while $C_\pm^2, C_\pm^3, C_\pm^4$ form two 3-cycles.

14.1.1. *Lifted facet decomposition and the double cover.* Let $\mu_8 : \pi_1(M_8) \rightarrow \{\pm 1\}$ denote the monodromy character determined by $\mu_8([m_e]) = -1$ on edge meridians, let $p_8 : \widetilde{M}_8 \rightarrow M_8$ be the associated connected double cover, and let τ_8 denote its nontrivial deck involution. For $i \in \{1, 2, 3, 4\}$ and $\epsilon \in \{+, -\}$, set

$$C_\epsilon^i := \pi(C_\epsilon^i) \subset S^3, \quad \mathring{C}_\epsilon^i := C_\epsilon^i \setminus \Gamma_8 \subset M_8.$$

The eight sets $\{\mathring{C}_\epsilon^i\}_{i,\epsilon}$ are the facets in M_8 determined by the cubical cell decomposition of S^3 obtained from the boundary complex of P_8 by radial projection. For $i \neq j$ and $\epsilon, \delta \in \{+, -\}$, let

$$F_{ij}^{\epsilon,\delta} := C_\epsilon^i \cap C_\delta^j, \quad \mathcal{F}_{ij}^{\epsilon,\delta} := \pi(F_{ij}^{\epsilon,\delta}) \subset S^3, \quad \mathring{\mathcal{F}}_{ij}^{\epsilon,\delta} := \mathcal{F}_{ij}^{\epsilon,\delta} \setminus \Gamma_8.$$

Then $\mathcal{F}_{ij}^{\epsilon,\delta}$ is the radial projection of a square face of ∂P_8 .

The preimage $p_8^{-1}(\mathring{C}_\epsilon^i)$ is the disjoint union of two copies of $\mathring{C}_\epsilon^i$; after fixing one choice of sheet we write

$$p_8^{-1}(\mathring{C}_\epsilon^i) = \mathring{C}_\epsilon^{i,+} \sqcup \mathring{C}_\epsilon^{i,-}, \quad \tau_8(\mathring{C}_\epsilon^{i,\pm}) = \mathring{C}_\epsilon^{i,\mp}.$$

Likewise,

$$p_8^{-1}(\mathcal{F}_{ij}^{\varepsilon,\delta,+}) = \mathcal{F}_{ij}^{\varepsilon,\delta,+} \sqcup \mathcal{F}_{ij}^{\varepsilon,\delta,-}, \quad \tau_8(\mathcal{F}_{ij}^{\varepsilon,\delta,\pm}) = \mathcal{F}_{ij}^{\varepsilon,\delta,\mp}.$$

Since Ad_q preserves Γ_8 and permutes its edges, Lemma 12.2 gives $\mu_8 \circ (\text{Ad}_q)_* = \mu_8$. Hence Lemma 12.3 gives a lift of Ad_q to $\widetilde{M}_8$, and the only two such lifts differ by composition with the deck involution τ_8 . Since Ad_q has order 3, one lift has order 3, and the other has order 6. Let $\widetilde{\text{Ad}}_q$ denote the order-3 lift. Then $\tau_8 \circ \widetilde{\text{Ad}}_q$ is the order-6 lift, characterized by $(\tau_8 \circ \widetilde{\text{Ad}}_q)^3 = \tau_8$. After choosing the sheet labels accordingly, its action on the lifted facets is

$$\begin{aligned} \tau_8 \widetilde{\text{Ad}}_q : & \quad (c_+^{1,+} \ c_+^{1,-}) (c_-^{1,+} \ c_-^{1,-}) (c_+^{2,+} \ c_+^{3,-} \ c_+^{4,+} \ c_+^{2,-} \ c_+^{3,+} \ c_+^{4,-}) \\ & \quad (c_-^{2,+} \ c_-^{3,-} \ c_-^{4,+} \ c_-^{2,-} \ c_-^{3,+} \ c_-^{4,-}) \end{aligned}$$

while the action of $\widetilde{\text{Ad}}_q$

$$\widetilde{\text{Ad}}_q : (c_+^{2,+} \ c_+^{3,+} \ c_+^{4,+}) (c_+^{2,-} \ c_+^{3,-} \ c_+^{4,-}) (c_-^{2,+} \ c_-^{3,+} \ c_-^{4,+}) (c_-^{2,-} \ c_-^{3,-} \ c_-^{4,-})$$

is illustrated in Fig. 6.

PROPOSITION 14.1 (Order-3 symmetries of the 8-cell). *Let $P_8 \subset \mathbb{R}^4$ be the regular 8-cell with vertices $\{V_1^{(8)}, \dots, V_{16}^{(8)}\} \subset \mathbb{R}^4$, and let $\Gamma_8 \subset S^3$ be the radial projection of the 1-skeleton of P_8 . Set $G_8 := \text{Sym}^+(P_8) \subset SO(4)$. Let $E_0^{(8)} := [V_1^{(8)}, V_9^{(8)}]$. For any edge $E^{(8)} = [V_i^{(8)}, V_j^{(8)}]$ of P_8 , there exists $h \in G_8$ with $h(E_0^{(8)}) = E^{(8)}$, and the two nontrivial order-3 elements of G_8 fixing $E^{(8)}$ pointwise are $h \text{Ad}_q h^{-1}$ and $(h \text{Ad}_q h^{-1})^{-1}$. The restriction of each to $W_{E^{(8)}}^\perp$, where $W_{E^{(8)}} := \text{span}_{\mathbb{R}}\{V_i^{(8)}, V_j^{(8)}\}$, is a rotation through angle $\pm 2\pi/3$. In particular, G_8 contains exactly 32 elements of order 3.*

PROOF. The coordinate formula $\text{Ad}_q(x_1, x_2, x_3, x_4) = (x_1, x_4, x_2, x_3)$ shows that $\text{Ad}_q \in G_8$ has order 3 and preserves $P_8 = [-1, 1]^4$. By quaternion conjugation, Ad_q fixes the 2-plane

$$F = \text{span}_{\mathbb{R}}\{1, i + j + k\}$$

pointwise and rotates $F^\perp$ by $2\pi/3$. Since $V_1^{(8)}$ and $V_9^{(8)}$ lie in F , the edge $E_0^{(8)} = [V_1^{(8)}, V_9^{(8)}]$ is fixed pointwise by Ad_q . Because $V_1^{(8)}$ and $V_9^{(8)}$ are linearly independent and F is a 2-plane, one has

$$F = \text{span}_{\mathbb{R}}\{V_1^{(8)}, V_9^{(8)}\} = W_{E_0^{(8)}}.$$

Therefore Ad_q restricting to $W_{E_0^{(8)}}^\perp$ is a rotation by $2\pi/3$.

Since P_8 is regular, G_8 acts transitively on its edges, so for any edge $E^{(8)}$ there exists $h \in G_8$ with $h(E_0^{(8)}) = E^{(8)}$. Then $h \text{Ad}_q h^{-1}$ fixes $E^{(8)}$ pointwise and restricts on $W_{E^{(8)}}^\perp$ to a rotation by $\pm 2\pi/3$.

To see that there are no further nontrivial order-3 elements fixing $E^{(8)}$ pointwise, note that the 8-cell has Schläfli symbol $\{4, 3, 3\}$, so the edge figure is a triangle. Hence, the orientation-preserving action fixing E pointwise is cyclic of order 3.

Now we show the rotation group G_8 of the 8-cell has exactly 32 elements of order 3. View G_8 as the orientation-preserving permutation group of P_8 , every $g \in G_8$ acts on the standard basis $\{e_1, e_2, e_3, e_4\}$ by $g(e_i) = \pm e_{\sigma(i)}$ for some permutation $\sigma \in S_4$. If g has order 3, then σ must have order dividing 3. Since $\sigma \neq \text{id}$, it follows that σ must be a 3-cycle. There are exactly 8 such 3-cycles in S_4 . To determine when $g^3 = \text{id}$, let a be the fixed index of the 3-cycle σ , and let $\{b, c, d\}$ be the three cyclically permuted indices. Then $g^3(e_a) = \varepsilon_a e_a$, so $\varepsilon_a = 1$. On the 3-cycle, $g^3(e_b) = \varepsilon_b \varepsilon_c \varepsilon_d e_b$, and similarly for e_c and e_d . Hence $g^3 = \text{id}$ if and only if $\varepsilon_a = 1, \varepsilon_b \varepsilon_c \varepsilon_d = 1$. There are exactly four choices for $(\varepsilon_b, \varepsilon_c, \varepsilon_d)$:

$$(+, +, +), \quad (+, -, -), \quad (-, +, -), \quad (-, -, +).$$

Because a 3-cycle has sign $+1$, these four sign choices have determinant $\varepsilon_a \varepsilon_b \varepsilon_c \varepsilon_d = 1$, so they define elements of G_8 . Thus, each of the eight underlying 3-cycles gives exactly four elements of G_8 of order 3, for a total of 32. $\square$

14.2. The lift group $\tilde{G}_8$ and the associated $\mathbb{Z}/2$ -extension. From Lemma 12.3 and Lemma 12.2, one knows that μ_8 is G_8 -invariant and every $g \in G_8$ admits a lift. Define the *group of lifts*

$$\tilde{G}_8 := \{\tilde{g} \in \text{Homeo}(\widetilde{M}_8) \mid \exists g \in G_8 \text{ such that } p_8 \circ \tilde{g} = g \circ p_8\}.$$

Then Lemma 12.4 gives a short exact sequence

$$1 \longrightarrow \langle \tau_8 \rangle \longrightarrow \tilde{G}_8 \xrightarrow{\pi} G_8 \longrightarrow 1.$$

Lemma 12.5 shows that for each $g \in G_8$, a choice of lift $\tilde{g} : \widetilde{M}_8 \rightarrow \widetilde{M}_8$ induces a fiberwise linear bundle automorphism $\hat{g} : \mathcal{I}_8 \rightarrow \mathcal{I}_8$, and replacing $\tilde{g}$ by $\tau_8 \circ \tilde{g}$ replaces $\hat{g}$ by $-\hat{g}$. Thus, a splitting of the extension above gives a choice of lifts whose induced bundle maps $\hat{g}$ satisfy $\widehat{gh} = \hat{g} \circ \hat{h}$ for all $g, h \in G_8$.

PROPOSITION 14.2 (Splitting for the 8-cell group). *The G_8 -action on M_8 lifts to a fiberwise linear action by bundle automorphisms: for each $g \in G_8$, there is a bundle automorphism $\hat{g} : \mathcal{I}_8 \rightarrow \mathcal{I}_8$ covering g , and $\widehat{gh} = \hat{g} \circ \hat{h}$ for all $g, h \in G_8$.*

PROOF. Let $H_8 := \ker(\mu_8)$. Since $\mu_8 \circ g_* = \mu_8$ for every $g \in G_8$, the connected double cover p_8 corresponding to H_8 is preserved by G_8 . Hence every $g \in G_8$ admits a lift to $\widetilde{M}_8$, unique up to composition with the deck involution τ_8 . By Lemma 12.4, there is a short exact sequence

$$1 \longrightarrow \langle \tau_8 \rangle \longrightarrow \tilde{G}_8 \xrightarrow{\pi} G_8 \longrightarrow 1.$$

It suffices to construct a splitting $\sigma : G_8 \rightarrow \tilde{G}_8$. Choose generators $\sigma_1, \sigma_2, \sigma_3 \in G_8$ by

$$\begin{aligned}\sigma_1(x_1, x_2, x_3, x_4) &= (-x_2, x_1, x_3, x_4), \\ \sigma_2(x_1, x_2, x_3, x_4) &= (x_3, x_1, x_2, x_4), \\ \sigma_3(x_1, x_2, x_3, x_4) &= (x_1, x_4, x_2, x_3) = \text{Ad}_q(x_1, x_2, x_3, x_4).\end{aligned}$$

These are orientation-preserving symmetries of $P_8 = [-1, 1]^4$. According to Eq. (11.2), the group representation is:

$$\begin{aligned}G_8 &\cong [4, 3, 3]^+ \\ &= \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^4 = \sigma_2^3 = \sigma_3^3 = (\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2 = 1 \rangle.\end{aligned}$$

We now choose lifts of $\sigma_1, \sigma_2, \sigma_3$ with the same orders in $\tilde{G}_8$.

Step 1: choose a lift of σ_1 with $\tilde{\sigma}_1^4 = 1$. The fixed-point set of σ_1 on S^3 is

$$\text{Fix}(\sigma_1|_{S^3}) = \{x_1 = x_2 = 0\} \cap S^3,$$

a great circle. This circle cannot be contained in Γ_8 ; otherwise σ_1 would fix the radial image of some edge pointwise, whereas the orientation-preserving pointwise stabilizer of an edge is cyclic of order 3 by the triangular edge figure. Hence $\text{Fix}(\sigma_1) \cap M_8 \neq \emptyset$. Choose $x_{\sigma_1} \in \text{Fix}(\sigma_1) \cap M_8$, $\tilde{x}_{\sigma_1} \in p_8^{-1}(x_{\sigma_1})$, and let $\tilde{\sigma}_1$ be the unique lift of σ_1 such that $\tilde{\sigma}_1(\tilde{x}_{\sigma_1}) = \tilde{x}_{\sigma_1}$. Then $\tilde{\sigma}_1^4$ is a deck transformation, since it covers $\sigma_1^4 = \text{id}$. Because $\tilde{\sigma}_1^4$ fixes $\tilde{x}_{\sigma_1}$ and the nontrivial deck involution τ_8 acts freely, it follows that $\tilde{\sigma}_1^4 = \text{id}$.

Step 2: choose lifts of σ_2 and σ_3 with $\tilde{\sigma}_2^3 = \tilde{\sigma}_3^3 = \text{id}$. By Lemma 12.3, σ_2 and σ_3 admit lifts $\tilde{\sigma}_2, \tilde{\sigma}_3$ to $\tilde{M}_8$. Since $\sigma_2^3 = \sigma_3^3 = \text{id}$, the elements $\tilde{\sigma}_2^3$ and $\tilde{\sigma}_3^3$ are deck transformations, so $\tilde{\sigma}_2^3, \tilde{\sigma}_3^3 \in \{\text{id}, \tau_8\}$. For $j = 2, 3$, choose the final lift of σ_j to be either the current lift $\tilde{\sigma}_j$ or the other lift $\tau_8\tilde{\sigma}_j$: keep $\tilde{\sigma}_j$ if $\tilde{\sigma}_j^3 = \text{id}$, and replace it by $\tau_8\tilde{\sigma}_j$ if $\tilde{\sigma}_j^3 = \tau_8$. Since every lift commutes with τ_8 , in the second case $(\tau_8\tilde{\sigma}_j)^3 = \tau_8^3\tilde{\sigma}_j^3 = \tau_8 \cdot \tau_8 = \text{id}$. After this possible replacement, we continue to denote the chosen lifts by $\tilde{\sigma}_2, \tilde{\sigma}_3$, and they satisfy $\tilde{\sigma}_2^3 = \tilde{\sigma}_3^3 = \text{id}$.

Step 3: verify the involution relations in $\tilde{G}_8$. A direct computation gives

$$\begin{aligned}\sigma_1\sigma_2(x_1, x_2, x_3, x_4) &= (-x_1, x_3, x_2, x_4), \\ \sigma_2\sigma_3(x_1, x_2, x_3, x_4) &= (x_2, x_1, x_4, x_3), \\ \sigma_1\sigma_2\sigma_3(x_1, x_2, x_3, x_4) &= (-x_1, x_2, x_4, x_3).\end{aligned}$$

Hence

$$\begin{aligned}\text{Fix}((\sigma_1\sigma_2)|_{S^3}) &= \{x_1 = 0, x_2 = x_3\} \cap S^3, \\ \text{Fix}((\sigma_2\sigma_3)|_{S^3}) &= \{x_1 = x_2, x_3 = x_4\} \cap S^3, \\ \text{Fix}((\sigma_1\sigma_2\sigma_3)|_{S^3}) &= \{x_1 = 0, x_3 = x_4\} \cap S^3.\end{aligned}$$

Each is a great circle. None is contained in Γ_8 ; otherwise, the corresponding involution would fix the radial image of some edge pointwise, impossible because the orientation-preserving pointwise stabilizer of an edge is cyclic of order 3. Hence, for each $g \in \{\sigma_1\sigma_2, \sigma_2\sigma_3, \sigma_1\sigma_2\sigma_3\}$, one has $\text{Fix}(g) \cap M_8 \neq \emptyset$.

Now set $\tilde{g}_{12} := \tilde{\sigma}_1 \tilde{\sigma}_2$, $\tilde{g}_{23} := \tilde{\sigma}_2 \tilde{\sigma}_3$, $\tilde{g}_{123} := \tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\sigma}_3$. These are lifts of $\sigma_1 \sigma_2$, $\sigma_2 \sigma_3$, and $\sigma_1 \sigma_2 \sigma_3$, respectively. Since

$$(\sigma_1 \sigma_2)^2 = (\sigma_2 \sigma_3)^2 = (\sigma_1 \sigma_2 \sigma_3)^2 = \text{id}$$

in G_8 , each of $\tilde{g}_{12}^2$, $\tilde{g}_{23}^2$, and $\tilde{g}_{123}^2$ is a deck transformation.

We claim that each of these deck transformations is the identity. Let $g \in \{\sigma_1 \sigma_2, \sigma_2 \sigma_3, \sigma_1 \sigma_2 \sigma_3\}$ and let $\tilde{g}$ denote the corresponding lift above. Choose $x \in \text{Fix}(g) \cap M_8$ and $\tilde{x} \in p_8^{-1}(x)$. Since $g(x) = x$, the point $\tilde{g}(\tilde{x})$ lies in the two-point fiber $p_8^{-1}(x)$, so $\tilde{g}(\tilde{x}) \in \{\tilde{x}, \tau_8(\tilde{x})\}$. Since $\tilde{g}$ is a lift, Lemma 12.5(1) gives $\tilde{g}\tau_8 = \tau_8\tilde{g}$. In either case, $\tilde{g}^2(\tilde{x}) = \tilde{x}$. But $\tilde{g}^2$ is a deck transformation, and the nontrivial deck involution τ_8 has no fixed points. Therefore $\tilde{g}^2 = \text{id}$. Applying this to $g = \sigma_1 \sigma_2, \sigma_2 \sigma_3, \sigma_1 \sigma_2 \sigma_3$ gives $(\tilde{\sigma}_1 \tilde{\sigma}_2)^2 = (\tilde{\sigma}_2 \tilde{\sigma}_3)^2 = (\tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\sigma}_3)^2 = \text{id}$.

Thus, the chosen lifts satisfy all defining relations of the presentation of G_8 . Therefore the assignment $\sigma_1 \mapsto \tilde{\sigma}_1, \sigma_2 \mapsto \tilde{\sigma}_2, \sigma_3 \mapsto \tilde{\sigma}_3$ extends to a homomorphism $s : G_8 \rightarrow \tilde{G}_8$ such that $\pi \circ s = \text{id}_{G_8}$. Hence, the extension splits.

Finally, for each $g \in G_8$, let $\hat{g} : \mathcal{I}_8 \rightarrow \mathcal{I}_8$ be the fiberwise linear bundle automorphism induced by the lift $s(g)$ via Lemma 12.5. Since s is a homomorphism, these bundle maps satisfy $\widehat{gh} = \hat{g} \circ \hat{h}$ for all $g, h \in G_8$. This proves the proposition. $\square$

15. Order-3 isometries on regular 24-cell

The goal of this section is to verify the 24-cell case of Propositions 11.11 and 11.12. Proposition 15.1 shows that for each edge of P_{24} there is an order-3 element of G_{24} fixing that edge pointwise and acting on the normal 2-plane by rotation through angle $\pm 2\pi/3$; this gives the 24-cell case of Proposition 11.11. The even-valency statement needed to define the monodromy character μ_{24} is supplied by Proposition 11.10. Proposition 15.2 then shows that the associated $\mathbb{Z}/2$ -extension splits, and hence that the resulting splitting gives a homomorphic choice of lifts to the double cover and induces a fiberwise linear action by bundle automorphisms on the associated line bundle.

15.1. The 24-cell model. A 24-cell P_{24} is the convex hull of the 24 unit Hurwitz quaternions in $\mathbb{H} \simeq \mathbb{R}^4$, which has 24 vertices, 96 edges, and 24 octahedral 3-cells. Its edge length is 1, and since $V_{24} \subset S^3$, every vertex has distance 1 from the origin. Let

$$\mathcal{V}_{24} := \{\pm 1, \pm i, \pm j, \pm k\} \cup \left\{ \frac{\pm 1 \pm i \pm j \pm k}{2} \right\} \subset S^3 \subset \mathbb{H},$$

We enumerate them lexicographically as $V_1^{(24)}, \dots, V_{24}^{(24)}$, where the 16 half-Hurwitz vertices are ordered by their sign vectors $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{\pm 1\}^4$

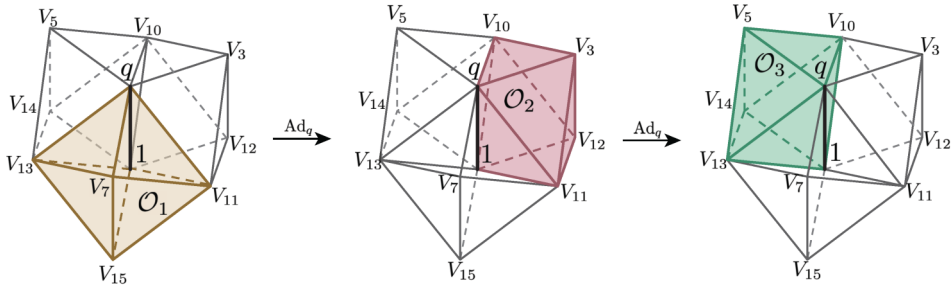


FIGURE 7. The three octahedral facets $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ of the 24-cell P_{24} meeting along the reference edge $E_0^{(24)} = [1, q]$, together with the induced facet dynamics under Ad_q .

with $+$ preceding $-$. Thus

$$V_1^{(24)} = 1, \quad V_8^{(24)} = -k, \quad V_9^{(24)} = \frac{1}{2}(1+i+j+k), \quad V_{24}^{(24)} = \frac{1}{2}(-1-i-j-k).$$

For $q := \frac{1}{2}(1+i+j+k) = V_9^{(24)}$, we obtain

$$\text{Ad}_q = (V_3 V_5 V_7)(V_4 V_6 V_8)(V_{10} V_{13} V_{11})(V_{12} V_{14} V_{15})(V_{18} V_{21} V_{19})(V_{20} V_{22} V_{23}),$$

and it fixes $V_1, V_2, V_9, V_{16}, V_{17}, V_{24}$. Two vertices $V_i^{(24)}$ and $V_j^{(24)}$ are joined by an edge if and only if $|V_i^{(24)} - V_j^{(24)}| = 1$. Define $P_{24} := \text{Hull}(\mathcal{V}_{24}) \subset \mathbb{R}^4$. This is the regular 24-cell. It has Schläfli symbol $\{3, 4, 3\}$; in particular, its facets are regular octahedra, its 2-faces are equilateral triangles, its vertex figures are cubes, and its edge figure is a triangle. Set $G_{24} := \text{Sym}^+(P_{24}) \subset SO(4)$.

Fig. 7 is a three-dimensional perspective schematic of the local boundary geometry of the 24-cell near the reference edge $E_0^{(24)} = [1, q]$: it shows the three octahedral facets $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ meeting along this edge and their cyclic permutation under Ad_q . To assign labels to Fig. 7, we first fix the reference edge $E_0^{(24)} = [1, q] = [V_1, V_9]$. Let $N(v)$ denote the set of vertices adjacent to v . The distance formulas give

$$N(V_1) = \{V_9, V_{10}, V_{11}, V_{12}, V_{13}, V_{14}, V_{15}, V_{16}\},$$

$$N(V_9) = \{V_1, V_3, V_5, V_7, V_{10}, V_{11}, V_{13}, V_{17}\},$$

and therefore $N(V_1) \cap N(V_9) = \{V_{10}, V_{11}, V_{13}\}$.

For the pair $\{V_{11}, V_{13}\}$, one finds

$$N(V_1) \cap N(V_{11}) \cap N(V_{13}) = \{V_9, V_{15}\}, \quad N(V_9) \cap N(V_{11}) \cap N(V_{13}) = \{V_1, V_7\}.$$

Thus the first octahedral facet through $E_0^{(24)}$ is $\mathcal{O}_1 = \{V_1, V_7, V_9, V_{11}, V_{13}, V_{15}\}$.

For the pair $\{V_{10}, V_{13}\}$, one finds

$$N(V_1) \cap N(V_{10}) \cap N(V_{13}) = \{V_9, V_{14}\}, \quad N(V_9) \cap N(V_{10}) \cap N(V_{13}) = \{V_1, V_5\}.$$

Thus the second octahedral facet through $E_0^{(24)}$ is $\mathcal{O}_2 = \{V_1, V_5, V_9, V_{10}, V_{13}, V_{14}\}$.

For the pair $\{V_{10}, V_{11}\}$, one finds

$$N(V_1) \cap N(V_{10}) \cap N(V_{11}) = \{V_9, V_{12}\}, \quad N(V_9) \cap N(V_{10}) \cap N(V_{11}) = \{V_1, V_3\}.$$

Thus the third octahedral facet through $E_0^{(24)}$ is $\mathcal{O}_3 = \{V_1, V_3, V_9, V_{10}, V_{11}, V_{12}\}$.

PROPOSITION 15.1 (Order-3 symmetries of the 24-cell). *Let $P_{24} \subset \mathbb{R}^4$ be a regular 24-cell, let $\Gamma_{24} \subset S^3$ be the radial projection of the 1-skeleton of P_{24} , and set $G_{24} := \text{Sym}^+(P_{24}) \subset SO(4)$. Let $E_0^{(24)} := [1, q]$. For any edge $E^{(24)} = [V_i^{(24)}, V_j^{(24)}]$ of P_{24} , there exists $h \in G_{24}$ with $h(E_0^{(24)}) = E^{(24)}$, and the two nontrivial order-3 elements of G_{24} fixing $E^{(24)}$ pointwise are $h \text{Ad}_q h^{-1}$ and its inverse. Each restricts on $W_{E^{(24)}}^\perp, W_{E^{(24)}} := \text{span}_{\mathbb{R}} \{V_i^{(24)}, V_j^{(24)}\}$ to a rotation by $\pm 2\pi/3$. In particular, G_{24} contains exactly 32 elements of order 3 fixing an edge pointwise.*

PROOF. We first show that $\text{Ad}_q \in G_{24}$. As $\text{Ad}_q(\mathcal{V}_{24}) = \mathcal{V}_{24}$ and $\text{Ad}_q \in SO(4)$, $\text{Ad}_q \in \text{Sym}^+(P_{24}) = G_{24}$. Also, $\text{Fix}(\text{Ad}_q) = \text{span}_{\mathbb{R}}\{1, i + j + k\}$ and Ad_q rotates $\text{Fix}^\perp(\text{Ad}_q)$ by angle $2\pi/3$. So Ad_q fixes the reference edge $E_0^{(24)}$ pointwise. Therefore Ad_q restricting on $W_{E_0^{(24)}}^\perp$ is a rotation by $2\pi/3$. Since P_{24} is regular, G_{24} acts transitively on its edges. Hence for any edge E there exists $h \in G_{24}$ with $h(E_0^{(24)}) = E$. Then $h \text{Ad}_q h^{-1}$ fixes E pointwise and restricts on $W_E^\perp$ to a rotation by $2\pi/3$, while its inverse restricts to a rotation by $-2\pi/3$, as illustrated in Fig. 7.

The 24-cell has 96 edges, and $\text{Fix}(\text{Ad}_q) \cap \mathcal{V}_{24} = \{\pm 1, \pm q, \pm q^2\}$. Let $u = (i + j + k)/\sqrt{3}$, so $q = \cos(\pi/3) + u \sin(\pi/3)$. On the circle $\text{Fix}(\text{Ad}_q) \cap S^3$, multiplication by q is rotation through angle $\pi/3$; hence the six points $q^0, \dots, q^5$, namely $1, q, q^2, -1, -q, -q^2$, occur in this cyclic order and form a regular hexagon. By the adjacency criterion, two vertices u and v are connected by an edge if and only if $\Re(u\bar{v}) = \frac{1}{2}$. So the edges of P_{24} contained in $\text{Fix}(\text{Ad}_q)$ are exactly the six sides of this hexagon. The same conclusion holds for every conjugate plane. Furthermore, if two such hexagons share an edge $E = [V_i^{(24)}, V_j^{(24)}]$, then their fixed planes both contain $W_E = \text{span}_{\mathbb{R}}\{V_i^{(24)}, V_j^{(24)}\}$. Since both fixed planes are two-dimensional, the fixed planes coincide, and hence the hexagons coincide. It follows that the 96 edges of P_{24} are partitioned into disjoint hexagons of size 6, and each corresponds to exactly two nontrivial order-3 elements fixing an edge pointwise, for a total of 32. $\square$

15.2. The lift group $\tilde{G}_{24}$ and the associated $\mathbb{Z}/2$ -extension. Let $M_{24} := S^3 \setminus \Gamma_{24}$, and let $\mu_{24} : \pi_1(M_{24}) \rightarrow \{\pm 1\}$ be the monodromy character determined by $\mu_{24}([m_e]) = -1$ for each edge meridian m_e , which is

well-defined by Proposition 11.10. Let $p_{24} : \widetilde{M}_{24} \rightarrow M_{24}$ be the associated connected double cover corresponding to $H_{24} := \ker(\mu_{24})$, let τ_{24} denote its nontrivial deck involution, and let $\mathcal{I}_{24} \rightarrow M_{24}$ be the associated real line bundle. Define

$$\widetilde{G}_{24} := \left\{ \widetilde{g} \in \text{Homeo}(\widetilde{M}_{24}) \mid \exists g \in G_{24} \text{ such that } p_{24} \circ \widetilde{g} = g \circ p_{24} \right\}.$$

PROPOSITION 15.2 (Splitting for the 24-cell group). *The G_{24} -action on M_{24} lifts to a fiberwise linear action by bundle automorphisms: for each $g \in G_{24}$, there is a bundle automorphism $\widehat{g} : \mathcal{I}_{24} \rightarrow \mathcal{I}_{24}$ covering g , such that*

$$\widehat{gh} = \widehat{g} \circ \widehat{h} \quad \text{for all } g, h \in G_{24}.$$

PROOF. By Proposition 11.10, each vertex of Γ_{24} has valency 8, so the assignment $\mu_{24}([m_e]) = -1$ on edge meridians is well-defined. Since every $g \in G_{24}$ preserves Γ_{24} and permutes its edges, Lemma 12.2 gives $\mu_{24} \circ g_* = \mu_{24}$. Hence $g_*(H_{24}) = H_{24}$, so every $g \in G_{24}$ admits a lift to $\widetilde{M}_{24}$, unique up to composition with the deck involution τ_{24} . It follows that

$$1 \longrightarrow \langle \tau_{24} \rangle \longrightarrow \widetilde{G}_{24} \xrightarrow{\pi} G_{24} \longrightarrow 1$$

is a short exact sequence, so it suffices to construct a splitting $\sigma : G_{24} \rightarrow \widetilde{G}_{24}$. *Step 1: choose lifts of σ_1 and σ_3 with $\widetilde{\sigma}_1^3 = \widetilde{\sigma}_3^3 = \text{id}$.* By Lemma 12.3, σ_1 and σ_3 admit lifts $\widetilde{\sigma}_1, \widetilde{\sigma}_3$ to $\widetilde{M}_{24}$. Since $\sigma_1^3 = \sigma_3^3 = \text{id}$, the elements $\widetilde{\sigma}_1^3$ and $\widetilde{\sigma}_3^3$ are deck transformations, $\widetilde{\sigma}_1^3, \widetilde{\sigma}_3^3 \in \{\text{id}, \tau_{24}\}$. If $\widetilde{\sigma}_1^3 = \tau_{24}$, replace $\widetilde{\sigma}_1$ by $\tau_{24}\widetilde{\sigma}_1$; similarly for $\widetilde{\sigma}_3$. Since every lift commutes with τ_{24} ,

$$(\tau_{24}\widetilde{\sigma}_1)^3 = \tau_{24}^3\widetilde{\sigma}_1^3 = \tau_{24} \cdot \tau_{24} = \text{id}, \quad (\tau_{24}\widetilde{\sigma}_3)^3 = \tau_{24}^3\widetilde{\sigma}_3^3 = \tau_{24} \cdot \tau_{24} = \text{id}.$$

Thus, we may choose lifts $\widetilde{\sigma}_1, \widetilde{\sigma}_3$ such that $\widetilde{\sigma}_1^3 = \widetilde{\sigma}_3^3 = \text{id}$.

Step 2: choose a lift of σ_2 of order 4. Following [17, Eq. (4), p. 1311], take the following standard generators of $[3, 4, 3]^+$:

$$\begin{aligned} \sigma_1(x_1, x_2, x_3, x_4) &= (x_1, x_4, x_2, x_3), \\ \sigma_2(x_1, x_2, x_3, x_4) &= (x_1, x_2, -x_4, x_3), \\ \sigma_3(x_1, x_2, x_3, x_4) &= \frac{1}{2}(x_1 + x_2 + x_3 + x_4, x_1 + x_2 - x_3 - x_4, \\ &\quad x_1 - x_2 + x_3 - x_4, -x_1 + x_2 + x_3 - x_4). \end{aligned}$$

Thus, $C := \text{Fix}(\sigma_2|_{S^3}) = \{x_3 = x_4 = 0\} \cap S^3$ is a great circle. We claim that $C \not\subset \Gamma_{24}$. Indeed, if $C \subset \Gamma_{24}$, then C contains the radial projection of some edge $E = [u, v]$ of P_{24} . Hence σ_2 fixes the edge E pointwise. Since the 24-cell has Schläfli symbol $\{3, 4, 3\}$, its edge figure is a triangle. Therefore, the orientation-preserving pointwise stabilizer of an edge acts faithfully on that triangle, so it is a subgroup of C_3 (rotation group of order 3). Hence, any nontrivial element fixing an edge pointwise has order 3, whereas σ_2 has order 4. Therefore, $C \cap M_{24} \neq \emptyset$. Choose $x_{\sigma_2} \in C \cap M_{24}$ and $\widetilde{x}_{\sigma_2} \in p_{24}^{-1}(x_{\sigma_2})$, and let $\widetilde{\sigma}_2$ be the unique lift of σ_2 such that $\widetilde{\sigma}_2(\widetilde{x}_{\sigma_2}) = \widetilde{x}_{\sigma_2}$. Then $\widetilde{\sigma}_2^4$ is a deck

transformation. Because $\tilde{\sigma}_2^4$ fixes $\tilde{x}_{\sigma_2}$, and the nontrivial deck involution acts freely, it follows that $\tilde{\sigma}_2^4 = \text{id}$.

Step 3: verify the involution relations upstairs. Set

$$\tilde{g}_{12} := \tilde{\sigma}_1 \tilde{\sigma}_2, \quad \tilde{g}_{23} := \tilde{\sigma}_2 \tilde{\sigma}_3, \quad \tilde{g}_{123} := \tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\sigma}_3.$$

These are lifts of $\sigma_1 \sigma_2$, $\sigma_2 \sigma_3$, and $\sigma_1 \sigma_2 \sigma_3$, respectively. Since $(\sigma_1 \sigma_2)^2 = (\sigma_2 \sigma_3)^2 = (\sigma_1 \sigma_2 \sigma_3)^2 = \text{id}$ in G_{24} , each of $\tilde{g}_{12}^2, \tilde{g}_{23}^2, \tilde{g}_{123}^2$ is a deck transformation. Let $g \in \{\sigma_1 \sigma_2, \sigma_2 \sigma_3, \sigma_1 \sigma_2 \sigma_3\}$, and let $\tilde{g}$ denote the corresponding lift above. We first show that $\text{Fix}(g) \cap M_{24} \neq \emptyset$. Since $g^2 = \text{id}$, it is enough to rule out $g = \pm \text{id}$. If $\sigma_1 \sigma_2 = \pm \text{id}$, then $\sigma_1 = \pm \sigma_2^{-1}$, contradicting $\sigma_1^3 = \text{id}$ and $\sigma_2^4 = \text{id}$; similarly, $\sigma_2 \sigma_3 \neq \pm \text{id}$. If $\sigma_1 \sigma_2 \sigma_3 = \pm \text{id}$, then $\sigma_1 \sigma_2 = \pm \sigma_3^{-1}$, which contradicts $(\sigma_1 \sigma_2)^2 = \text{id}$ and $\sigma_3^3 = \text{id}$. Thus $g \neq \pm \text{id}$, so by the same eigenvalue argument as in Proposition 13.2, $C_g := \text{Fix}(g|_{S^3})$ is a great circle. If $C_g \subset \Gamma_{24}$, then C_g contains the radial projection of some edge $E = [u, v]$ of P_{24} . But no element of order 2 fixes an edge pointwise, because the edge figure of the 24-cell is a triangle, so the orientation-preserving pointwise stabilizer of an edge is cyclic of order 3. This contradiction shows that $C_g \cap M_{24} \neq \emptyset$. Choose $x \in C_g \cap M_{24}$ and $\tilde{x} \in p_{24}^{-1}(x)$, then $\tilde{g}(\tilde{x})$ lies in the two-point fiber $p_{24}^{-1}(x) = \{\tilde{x}, \tau_{24}(\tilde{x})\}$. In either case, $\tilde{g}^2(\tilde{x}) = \tilde{x}$. Since $\tilde{g}^2$ is a deck transformation and the nontrivial deck involution τ_{24} acts freely, it follows that $\tilde{g}^2 = \text{id}$. Thus

$$(\tilde{\sigma}_1 \tilde{\sigma}_2)^2 = (\tilde{\sigma}_2 \tilde{\sigma}_3)^2 = (\tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\sigma}_3)^2 = \text{id}.$$

Hence, the chosen lifts satisfy all defining relations of the presentation; therefore, the assignments $\sigma_1 \mapsto \tilde{\sigma}_1, \sigma_2 \mapsto \tilde{\sigma}_2, \sigma_3 \mapsto \tilde{\sigma}_3$ extend to a homomorphism $\sigma : G_{24} \rightarrow \tilde{G}_{24}$ with $\pi \circ \sigma = \text{id}_{G_{24}}$. Hence, the extension splits. For each $g \in G_{24}$, let $\hat{g} : \mathcal{I}_{24} \rightarrow \mathcal{I}_{24}$ be the fiberwise linear bundle automorphism induced by the lift $\sigma(g)$ via Lemma 12.5. Since σ is a homomorphism, these satisfy $\widehat{gh} = \hat{g} \circ \hat{h}$ for all $g, h \in G_{24}$. This proves the proposition. $\square$

16. Order-3 isometries on regular 120-cell

The goal of this section is to verify the 120-cell case of Propositions 11.12 and 11.11. Using the model in §16.1, Proposition 16.2 supplies the odd-order pointwise stabilizers of edges required for Proposition 11.11. Proposition 16.3 then proves that the associated $\mathbb{Z}/2$ -extension splits.

16.1. The 120-cell model. A 120-cell is the regular convex 4-polytope with Schläfli symbol $\{5, 3, 3\}$. Its facets are regular dodecahedra, its 2-faces are regular pentagons, its vertex figures are regular tetrahedra, and its edge figure is a triangle. P_{120} has 600 vertices, 1200 edges, and 120 dodecahedral 3-cells, denoted $\mathcal{D}_i$ as depicted in Fig. 8. Its edge length is $3 - \sqrt{5}$, and every listed vertex has norm $2\sqrt{2}$, so every vertex has distance $2\sqrt{2}$ from the origin. Let $\phi := \frac{1+\sqrt{5}}{2}$. We use the standard coordinate model of

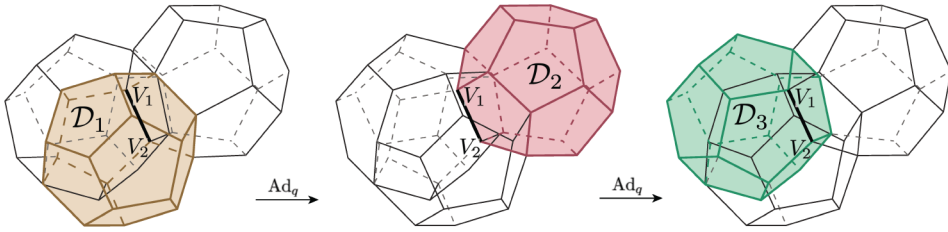


FIGURE 8. The three dodecahedral facets $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ of the 120-cell P_{120} meeting along the reference edge $E_0^{(120)} = [V_1^{(120)}, V_2^{(120)}]$, together with the induced facet dynamics under Ad_q .

the regular 120-cell, whose 600 vertices are the union of the following seven families:

- Family 1: $2(\pm 1, \pm 1, 0, 0)$ and all permutations,
- Family 2: $(\pm\sqrt{5}, \pm 1, \pm 1, \pm 1)$ and all permutations,
- Family 3: $(\pm\phi^{-2}, \pm\phi, \pm\phi, \pm\phi)$ and all permutations,
- Family 4: $(\pm\phi^2, \pm\phi^{-1}, \pm\phi^{-1}, \pm\phi^{-1})$ and all permutations,
- Family 5: $(\pm\phi^2, \pm\phi^{-2}, \pm 1, 0)$ and all even permutations,
- Family 6: $(\pm\sqrt{5}, \pm\phi^{-1}, \pm\phi, 0)$ and all even permutations,
- Family 7: $(\pm 2, \pm 1, \pm\phi, \pm\phi^{-1})$ and all even permutations.

As Ad_q fixes the first coordinate and cyclically permutes the last three, Ad_q preserves the seven-family vertex set above: Families 1–4 are closed under all permutations, and Families 5–7 are closed under even permutations. Thus $\text{Ad}_q \in G_{120}$.

Let $V_{120} := \{V_1^{(120)}, \dots, V_{600}^{(120)}\} \subset \mathbb{R}^4 \simeq \mathbb{H}$ denote this set, and define $P_{120} := \text{Hull}(V_{120}) \subset \mathbb{R}^4$. Let $\Gamma_{120} \subset S^3$ be the radial projection of the 1-skeleton of P_{120} , set $M_{120} := S^3 \setminus \Gamma_{120}$, and write $G_{120} := \text{Sym}^+(P_{120}) \subset SO(4)$.

LEMMA 16.1. *Let $e = [u, v]$ be an edge of the regular 120-cell. Then the plane $P_e := \text{span}\{u, v\}$ contains exactly 12 vertices of the 120-cell and exactly 6 edges of the 120-cell.*

PROOF. Since the 120-cell is regular, its symmetry group is transitive on edges. Hence, it is enough to prove the claim for one specific edge. Take $u = (2, 2, 0, 0), v = (\sqrt{5}, \phi, 0, \phi^{-1})$. A direct computation gives $\|u - v\|^2 = (3 - \sqrt{5})^2$, so $[u, v]$ is an edge. Thus

$$\text{span}\{u, v\} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_3 = 0, x_1 = x_2 + x_4\}.$$

Family 1 contributes exactly the six vertices

$$A_0 = (2, 2, 0, 0), \quad A_1 = (2, 0, 0, 2), \quad A_2 = (0, -2, 0, 2),$$

$$A_3 = (-2, -2, 0, 0), \quad A_4 = (-2, 0, 0, -2), \quad A_5 = (0, 2, 0, -2).$$

Family 6 contributes exactly the six vertices

$$\begin{aligned} B_0 &= (\sqrt{5}, \phi, 0, \phi^{-1}), \quad B_1 = (\phi, -\phi^{-1}, 0, \sqrt{5}), \quad B_2 = (-\phi^{-1}, -\sqrt{5}, 0, \phi), \\ B_3 &= (-\sqrt{5}, -\phi, 0, -\phi^{-1}), \quad B_4 = (-\phi, \phi^{-1}, 0, -\sqrt{5}), \quad B_5 = (\phi^{-1}, \sqrt{5}, 0, -\phi). \end{aligned}$$

Substituting the vertices from Families 2, 3, 4, 5, and 7 into $x_3 = 0$ and $x_1 = x_2 + x_4$ gives no solutions, so these families contribute no vertices to P_e . Therefore, P_e contains exactly 12 vertices of the 120-cell. A direct computation shows that the distance $3 - \sqrt{5}$ occurs exactly for the six pairs: $A_0B_0, A_1B_1, A_2B_2, A_3B_3, A_4B_4, A_5B_5$. $\square$

PROPOSITION 16.2 (Order-3 symmetries of the 120-cell). *Let $P_{120} \subset \mathbb{R}^4$ be a regular 120-cell. Let $\Gamma_{120} \subset S^3$ be the radial projection of its 1-skeleton, and set $G_{120} := \text{Sym}^+(P_{120}) \subset SO(4)$. Let $V_1^{(120)} := (\sqrt{5}, 1, 1, 1)$, $V_2^{(120)} := (\phi^2, \phi^{-1}, \phi^{-1}, \phi^{-1})$, and $E_0^{(120)} := [V_1^{(120)}, V_2^{(120)}]$. For any edge $E_{ij}^{(120)} := [V_i^{(120)}, V_j^{(120)}]$ of P_{120} , there exists $h \in G_{120}$ with $h(E_0^{(120)}) = E_{ij}^{(120)}$, and the two nontrivial order-3 elements of G_{120} fixing $E_{ij}^{(120)}$ pointwise are $h \text{Ad}_q h^{-1}$ and its inverse. Each restricts on $W_{E_{ij}^{(120)}}^\perp, W_{E_{ij}^{(120)}} := \text{span}_{\mathbb{R}} \{V_i^{(120)}, V_j^{(120)}\}$, to a rotation by $\pm 2\pi/3$. The number of elements of order 3 in G_{120} that fix some edge pointwise is 400.*

PROOF. The map $\text{Ad}_q \in G_{120}$ has order 3, fixes the 2-plane $F = \text{span}_{\mathbb{R}}\{1, i+j+k\}$ pointwise, and rotates $F^\perp$ by angle $2\pi/3$. Both $V_1^{(120)}$ and $V_2^{(120)}$ belong to the vertex set of the standard model, and both are fixed by Ad_q , since their last three coordinates are equal. Moreover, $\|V_1^{(120)} - V_2^{(120)}\|^2 = (3 - \sqrt{5})^2$. Hence $E_0^{(120)}$ is an edge of P_{120} . Since $V_1^{(120)}, V_2^{(120)} \in F$ are linearly independent, we have $W_{E_0^{(120)}} := \text{span}_{\mathbb{R}}\{V_1^{(120)}, V_2^{(120)}\} = F$. Therefore, Ad_q fixes $E_0^{(120)}$ pointwise and restricts on $W_{E_0^{(120)}}^\perp$ to a rotation by angle $2\pi/3$, as illustrated in Fig. 8.

Since the 120-cell is regular, G_{120} acts transitively on its edges. Hence, for any edge

$$E_{ij}^{(120)} := [V_i^{(120)}, V_j^{(120)}]$$

of P_{120} , there exists $h \in G_{120}$ with $h(E_0^{(120)}) = E_{ij}^{(120)}$. Then $h \text{Ad}_q h^{-1}$ fixes $E_{ij}^{(120)}$ pointwise and restricts on $W_{E_{ij}^{(120)}}^\perp$, with $W_{E_{ij}^{(120)}} := \text{span}_{\mathbb{R}}\{V_i^{(120)}, V_j^{(120)}\}$, to a rotation by $2\pi/3$, while its inverse restricts to a rotation by $-2\pi/3$.

The 120-cell has 1200 edges [5]. By Lemma 16.1, each edge lies in a unique plane through the origin, and each such plane contains exactly 6 edges. Therefore, the number of such planes is 200. There are exactly two

nontrivial rotations of order 3 fixing the plane P_e pointwise and rotating $P_e^\perp$ by $\pm 2\pi/3$. Conversely, if an element of order 3 fixes an edge $e = [u, v]$ pointwise, then it fixes the plane $\text{span}\{u, v\}$ pointwise, and therefore is one of the two order-3 rotations associated with that plane. Thus, the number of order-3 elements fixing some edge pointwise is 400. $\square$

16.2. The lift group $\tilde{G}_{120}$ and the associated $\mathbb{Z}/2$ -extension. By Proposition 11.10, each vertex of Γ_{120} has valency 4, so the assignment $\mu_{120}([m_e]) = -1$ on edge meridians is well-defined. Let $p_{120} : \tilde{M}_{120} \rightarrow M_{120}$ be the associated connected double cover corresponding to $H_{120} := \ker(\mu_{120})$, let τ_{120} denote its nontrivial deck involution, and let $\mathcal{I}_{120} \rightarrow M_{120}$ be the associated real line bundle. Define

$$\tilde{G}_{120} := \left\{ \tilde{g} \in \text{Homeo}(\tilde{M}_{120}) \mid \exists g \in G_{120} \text{ such that } p_{120} \circ \tilde{g} = g \circ p_{120} \right\}.$$

PROPOSITION 16.3 (Splitting for the 120-cell group). *The G_{120} -action on M_{120} lifts to a fiberwise linear action by bundle automorphisms $\hat{g} : \mathcal{I}_{120} \rightarrow \mathcal{I}_{120}$ covering $g \in G_{120}$, such that $\widehat{gh} = \hat{g} \circ \hat{h}$ for all $g, h \in G_{120}$.*

PROOF. Since every $g \in G_{120}$ preserves Γ_{120} and permutes its edges, Lemma 12.2 gives $\mu_{120} \circ g_* = \mu_{120}$. Hence, by Lemma 12.3, every $g \in G_{120}$ admits a lift to $\tilde{M}_{120}$, unique up to composition with τ_{120} . By Lemma 12.4, one therefore has a short exact sequence

$$1 \longrightarrow \langle \tau_{120} \rangle \longrightarrow \tilde{G}_{120} \xrightarrow{\pi} G_{120} \longrightarrow 1.$$

It suffices to construct a splitting $s : G_{120} \rightarrow \tilde{G}_{120}$.

Choose generators $\sigma_1, \sigma_2, \sigma_3 \in G_{120}$ as in Equation (11.4) with presentation

$$G_{120} \cong [5, 3, 3]^+ \\ = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^5 = \sigma_2^3 = \sigma_3^3 = (\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2 = 1 \rangle.$$

Set $g_{12} := \sigma_1\sigma_2$, $g_{23} := \sigma_2\sigma_3$, and $g_{123} := \sigma_1\sigma_2\sigma_3$. We show below that each of g_{12}, g_{23}, g_{123} has fixed-point set meeting M_{120} , which is enough to verify the involution relations upstairs.

Step 1: choose lifts of $\sigma_1, \sigma_2, \sigma_3$ with the same odd orders upstairs. By Lemma 12.3, each σ_i admits a lift to $\tilde{M}_{120}$. Choose arbitrary lifts $\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3$. Since $\sigma_1^5 = \sigma_2^3 = \sigma_3^3 = 1$, the elements $\tilde{\sigma}_1^5, \tilde{\sigma}_2^3, \tilde{\sigma}_3^3 \in \{\text{id}, \tau_{120}\}$. If $\tilde{\sigma}_1^5 = \tau_{120}$, replace $\tilde{\sigma}_1$ by $\tau_{120}\tilde{\sigma}_1$. Since every lift commutes with τ_{120} , $(\tau_{120}\tilde{\sigma}_1)^5 = \tau_{120}^5\tilde{\sigma}_1^5 = \tau_{120} \cdot \tau_{120} = \text{id}$. The same odd-order adjustment applies to $\tilde{\sigma}_2$ and $\tilde{\sigma}_3$. Thus we may assume $\tilde{\sigma}_1^5 = \tilde{\sigma}_2^3 = \tilde{\sigma}_3^3 = \text{id}$.

Step 2: verify the involution relations upstairs. Set

$$\tilde{g}_{12} := \tilde{\sigma}_1\tilde{\sigma}_2, \quad \tilde{g}_{23} := \tilde{\sigma}_2\tilde{\sigma}_3, \quad \tilde{g}_{123} := \tilde{\sigma}_1\tilde{\sigma}_2\tilde{\sigma}_3.$$

These are lifts of

$$g_{12} := \sigma_1\sigma_2, \quad g_{23} := \sigma_2\sigma_3, \quad g_{123} := \sigma_1\sigma_2\sigma_3.$$

Since $g_{12}^2 = g_{23}^2 = g_{123}^2 = 1$ in G_{120} , each of $\tilde{g}_{12}^2, \tilde{g}_{23}^2, \tilde{g}_{123}^2$ is a deck transformation.

We first claim that each of g_{12}, g_{23}, g_{123} is a nontrivial involution in $SO(4)$. Indeed, $g_{12} \neq \pm \text{id}$, for otherwise $\sigma_1 = \pm \sigma_2^{-1}$, impossible since σ_1 has order 5 while $\pm \sigma_2^{-1}$ has order 3 or 6. Likewise, $g_{23} \neq \pm \text{id}$, for otherwise $g_{123} = \pm \sigma_1$, contradicting $g_{123}^2 = 1$ and $|\sigma_1| = 5$. Finally, $g_{123} \neq \pm \text{id}$, for otherwise $g_{12} = \pm \sigma_3^{-1}$, contradicting $g_{12}^2 = 1$ and $|\sigma_3| = 3$.

Hence, for each $g \in \{g_{12}, g_{23}, g_{123}\}$, the fixed-point set $\text{Fix}(g|_{S^3})$ is a great circle. We claim that $\text{Fix}(g) \cap M_{120} \neq \emptyset$. If not, then $\text{Fix}(g) \subset \Gamma_{120}$. Since $\text{Fix}(g)$ is a great circle contained in the radial graph, it contains the radial image of some edge of P_{120} , so g fixes that edge pointwise. But the 120-cell has triangular edge figure, so the orientation-preserving pointwise stabilizer of an edge is cyclic of order 3, as in the proof of Proposition 11.11. In particular, no involution can fix an edge pointwise.

Therefore, we may choose $x_{12} \in \text{Fix}(g_{12}) \cap M_{120}$, $x_{23} \in \text{Fix}(g_{23}) \cap M_{120}$, $x_{123} \in \text{Fix}(g_{123}) \cap M_{120}$, and lifts $\tilde{x}_{12} \in p_{120}^{-1}(x_{12})$, $\tilde{x}_{23} \in p_{120}^{-1}(x_{23})$, $\tilde{x}_{123} \in p_{120}^{-1}(x_{123})$. Because each g fixes the corresponding point, the lift $\tilde{g}$ sends the chosen lift $\tilde{x}$ either to $\tilde{x}$ or to $\tau_{120}(\tilde{x})$. Since $\tilde{g}$ commutes with τ_{120} , in both cases $\tilde{g}^2(\tilde{x}) = \tilde{x}$. Since $\tilde{g}^2$ is a deck transformation and the nontrivial deck involution acts freely, it follows that $(\tilde{\sigma}_1 \tilde{\sigma}_2)^2 = (\tilde{\sigma}_2 \tilde{\sigma}_3)^2 = (\tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\sigma}_3)^2 = \text{id}$.

Thus, the chosen lifts satisfy all defining relations of the presentation of G_{120} . Therefore, the assignment $\sigma_1 \mapsto \tilde{\sigma}_1, \sigma_2 \mapsto \tilde{\sigma}_2, \sigma_3 \mapsto \tilde{\sigma}_3$ extends to a homomorphism $s : G_{120} \rightarrow \tilde{G}_{120}$ with $\pi \circ s = \text{id}_{G_{120}}$. Hence, the extension splits. For each $g \in G_{120}$, let $\hat{g} : \mathcal{I}_{120} \rightarrow \mathcal{I}_{120}$ be the fiberwise linear bundle automorphism induced by the lift $s(g)$ via Lemma 12.5. Since s is a homomorphism, these satisfy $\widehat{gh} = \hat{g} \circ \hat{h}$ for all $g, h \in G_{120}$. This proves the proposition. $\square$

17. Order-5 isometries on regular 600-cell

The goal of this section is to verify the 600-cell case of Propositions 11.12 and 11.11. Using the model in §17.1, Proposition 17.2 supplies the odd-order pointwise stabilizers of edges required for Proposition 11.11. Proposition 17.4 then proves that the associated $\mathbb{Z}/2$ -extension splits.

17.1. The 600-cell model. A 600-cell is the regular convex 4-polytope with Schläfli symbol $\{3, 3, 5\}$. Its facets are regular tetrahedra, its 2-faces are equilateral triangles, its vertex figures are regular icosahedra, and its edge figure is a pentagon. P_{600} has 120 vertices, 720 edges, and 600 tetrahedral 3-cells, denoted $\mathcal{F}_i$ as depicted in Fig. 9. Its edge length is ϕ^{-1} , and since $V_{600} \subset S^3$, every vertex has distance 1 from the origin. Explicitly, the vertices are the union of the following three families:

$$A_1 = \{(\pm 1, 0, 0, 0) \text{ and all permutations}\},$$

$$A_2 = \left\{ \frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1) \right\},$$

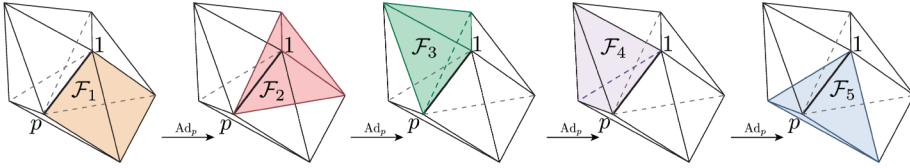


FIGURE 9. The five tetrahedral facets $\mathcal{F}_1, \dots, \mathcal{F}_5$ of the 600-cell P_{600} meeting along the reference edge $E_0^{(600)} = [1, p]$, together with the induced 5-cycle of facets under Ad_p .

$$A_3 = \left\{ \frac{1}{2}(0, \pm 1, \pm \phi, \pm \phi^{-1}) \text{ and all even permutations} \right\},$$

where $\phi := \frac{1+\sqrt{5}}{2}$. These three families contribute 8, 16, and 96 vertices, respectively. Let $\mathcal{V}_{600} := \{V_1^{(600)}, \dots, V_{120}^{(600)}\} \subset S^3$ denote this set, and define $P_{600} := \text{Hull}(\mathcal{V}_{600}) \subset \mathbb{R}^4$. Let $\Gamma_{600} \subset S^3$ be the radial projection of the 1-skeleton of P_{600} , set $M_{600} := S^3 \setminus \Gamma_{600}$, and write $G_{600} := \text{Sym}^+(P_{600}) \subset SO(4)$.

The set $\mathcal{V}_{600}$ is the binary icosahedral group $2I \subset S^3$, so it is closed under quaternion multiplication and inversion. Set

$$p := \frac{1}{2}(\phi + \phi^{-1}i + k).$$

Then $p \in A_3 \subset \mathcal{V}_{600}$, because its coordinate vector $(\phi, \phi^{-1}, 0, 1)$ is the even permutation of $(0, 1, \phi, \phi^{-1})$. Since $p \in S^3$, its inverse is its quaternionic conjugate:

$$p^{-1} = \bar{p} = \frac{1}{2}(\phi - \phi^{-1}i - k).$$

For $x \in \mathbb{H}$, define $\text{Ad}_p(x) := p x p^{-1}$. Direct computation gives

$$\begin{aligned} \text{Ad}_p(1) &= 1, \quad \text{Ad}_p(i) = \frac{1}{2}(i + \phi j + \phi^{-1}k), \quad \text{Ad}_p(j) = \frac{1}{2}(-\phi i + \phi^{-1}j + k), \\ \text{Ad}_p(k) &= \frac{1}{2}(\phi^{-1}i - j + \phi k). \end{aligned}$$

By $\mathbb{R}$ -linearity, $\text{Ad}_p(\pm 1) = \pm 1 \in A_1$, $\text{Ad}_p(\pm i), \text{Ad}_p(\pm j), \text{Ad}_p(\pm k) \in A_3$ because the coordinate vectors

$$(0, 1, \phi, \phi^{-1}), \quad (0, -\phi, \phi^{-1}, 1), \quad (0, \phi^{-1}, -1, \phi)$$

are all even permutations of $(0, \pm 1, \pm \phi, \pm \phi^{-1})$.

Let $x \in \mathcal{V}_{600}$. Since $\mathcal{V}_{600} = 2I$ is a group, and both p and p^{-1} lie in $\mathcal{V}_{600}$, the product $p x p^{-1}$ also lies in $\mathcal{V}_{600}$. To show injectivity, suppose $x, y \in \mathcal{V}_{600}$ and $\text{Ad}_p(x) = \text{Ad}_p(y)$. Then $p x p^{-1} = p y p^{-1}$. Multiplying on the left by p^{-1} and on the right by p , we get $x = y$. Hence Ad_p is injective on $\mathcal{V}_{600}$. Since $\text{Ad}_p(\mathcal{V}_{600}) \subseteq \mathcal{V}_{600}$ and $\mathcal{V}_{600}$ is finite, every injective self-map of $\mathcal{V}_{600}$ is bijective (also Ad_p has explicit inverse $\text{Ad}_{p^{-1}}$). Thus Ad_p preserves $P_{600} = \text{Hull}(\mathcal{V}_{600})$; since $\text{Ad}_p \in SO(4)$, it lies in G_{600} . The lemma below records

the vertices and edges of P_{600} contained in the fixed plane $\text{span}_{\mathbb{R}}\{1, \Im(p)\}$, which is used to verify the order-5 pointwise stabilizers.

LEMMA 17.1. *Let $\phi := \frac{1+\sqrt{5}}{2}$, $p := \frac{1}{2}(\phi + \phi^{-1}i + k) \in \mathcal{V}_{600} \subset S^3$, and $F := \text{span}_{\mathbb{R}}\{1, \Im(p)\} \subset \mathbb{H} \simeq \mathbb{R}^4$. Then $F \cap \mathcal{V}_{600} = \{1, p, p^2, p^3, p^4, -1, -p, -p^2, -p^3, -p^4\}$. These ten vertices lie on the unit circle $F \cap S^3$ as a regular decagon. Moreover, the edges of P_{600} contained in F are exactly the ten sides of this decagon.*

PROOF. Let $u = \text{Im}(p)/|\text{Im}(p)|$. Since $\text{span}_{\mathbb{R}}\{1, u\}$ is closed under multiplication, every power $p^n \in \text{span}_{\mathbb{R}}\{1, u\}$. Since $p = \cos(\pi/5) + u \sin(\pi/5)$, one has $p^5 = -1$, so these are ten distinct points. We now check $F \cap \mathcal{V}_{600}$ consists of exactly these ten vertices. A quaternion $x = a + bi + cj + dk$ lies in F if and only if $c = 0, d = \phi b$. Equivalently, $F = \{a + bi + \phi b k : a, b \in \mathbb{R}\}$.

- A_1 : The condition $c = 0$ and $d = \phi b$ restricts to $A_1 \cap F = \{\pm 1\}$.
- A_2 : Every such point has j -coordinate $\pm \frac{1}{2} \neq 0$, so none lies in F . Hence $A_2 \cap F = \emptyset$.
- A_3 : Among even permutations of $(0, 1, \phi, \phi^{-1})$, the ones with 0 in the third slot are:

$$(1, \phi, 0, \phi^{-1}), \quad (\phi, \phi^{-1}, 0, 1), \quad (\phi^{-1}, 1, 0, \phi).$$

The latter two satisfy $d = \phi b$. Therefore the only A_3 vertices in F are the 8 points:

$$\frac{1}{2}(\varepsilon\phi + \delta\phi^{-1}i + \delta k), \quad \frac{1}{2}(\varepsilon\phi^{-1} + \delta i + \delta\phi k), \quad \varepsilon, \delta \in \{\pm 1\}.$$

The eight A_3 -vertices listed above are precisely $p, p^2, p^3, p^4, -p, -p^2, -p^3, -p^4$. Together with $A_1 \cap F = \{\pm 1\}$, this gives the stated equality $F \cap \mathcal{V}_{600} = \{1, p, p^2, p^3, p^4, -1, -p, -p^2, -p^3, -p^4\}$. Since $p = \cos(\pi/5) + u \sin(\pi/5)$, these ten points are equally spaced on the circle $F \cap S^3$, hence form a regular decagon.

Finally, the edge length of P_{600} is ϕ^{-1} , and $\|p^{m+1} - p^m\| = \|1 - p\| = \phi^{-1}$. So consecutive vertices in the decagon are joined by edges of P_{600} . For any integers m, r ,

$$\|p^{m+r} - p^m\| = \|p^m(p^r - 1)\| = \|p^m\| \|p^r - 1\| = \|p^r - 1\| = 2 \left| \sin \frac{r\pi}{10} \right|.$$

If r is chosen modulo 10 with $0 \leq |r| \leq 5$, this simplifies to $\|p^{m+r} - p^m\| = 2 \sin\left(\frac{|r|\pi}{10}\right) > \phi^{-1}$ if $|r| \geq 2$. So no non-consecutive pair is joined by an edge. Therefore, the edges of P_{600} contained in F are exactly the ten sides of the regular decagon. $\square$

PROPOSITION 17.2 (Order-5 symmetries of the 600-cell). *Let $P_{600} \subset \mathbb{R}^4$ be the regular 600-cell, let $\Gamma_{600} \subset S^3$ be the radial projection of its 1-skeleton, and set $G_{600} := \text{Sym}^+(P_{600}) \subset SO(4)$. Let $E_0^{(600)} := [1, p]$. For any edge $E^{(600)} = [V_i^{(600)}, V_j^{(600)}]$ of P_{600} , there exists $h \in G_{600}$ with $h(E_0^{(600)}) =$*

$E^{(600)}$. Moreover, the four nontrivial order-5 elements of G_{600} fixing $E^{(600)}$ pointwise are $(h \operatorname{Ad}_p h^{-1})^k$, $k = 1, 2, 3, 4$. Each restricts to $W_{E^{(600)}}^\perp$, where $W_{E^{(600)}} := \operatorname{span}_{\mathbb{R}}\{V_i^{(600)}, V_j^{(600)}\}$, to a rotation by angle $2\pi k/5$. In particular, the orientation-preserving pointwise stabilizer of $E^{(600)}$ is cyclic of order 5.

PROOF. By the quaternionic description from §11.1, Ad_p fixes pointwise the 2-plane

$$F := \operatorname{span}_{\mathbb{R}}\{1, \Im(p)\}$$

and rotates $F^\perp$ by angle $\frac{2\pi}{5}$. By Lemma 17.1, the segment $E_0^{(600)} = [1, p]$ is an edge of P_{600} , and it is contained in F . Therefore, Ad_p fixes $E_0^{(600)}$ pointwise and restricts on $W_{E_0^{(600)}}^\perp$ to a rotation by angle $2\pi/5$, as illustrated in Fig. 9. Since P_{600} is regular, G_{600} acts transitively on its edges. Hence for any edge $E^{(600)}$ there exists $h \in G_{600}$ such that $h(E_0^{(600)}) = E^{(600)}$. Then $h \operatorname{Ad}_p h^{-1}$ fixes $E^{(600)}$ pointwise and acts on $W_{E^{(600)}}^\perp$ by rotation through $2\pi/5$. Its powers $(h \operatorname{Ad}_p h^{-1})^k$, $k = 1, 2, 3, 4$, act by rotations through angles $2\pi k/5$. Since the 600-cell has Schläfli symbol $\{3, 3, 5\}$, its edge figure is a pentagon, so any orientation-preserving symmetry fixing $E^{(600)}$ pointwise acts faithfully on this pentagon and hence lies in its rotation group C_5 ; the subgroup just constructed has order 5, so the orientation-preserving pointwise stabilizer of $E^{(600)}$ is exactly cyclic of order 5. $\square$

COROLLARY 17.3. *The number of order 5 elements in G_{600} preserving an edge of P_{600} is 288.*

PROOF. The 600-cell has 720 edges. By Proposition 17.2, the orientation-preserving pointwise stabilizer of any edge is cyclic of order 5. Therefore, each edge is preserved by exactly four nontrivial elements of order 5. By Proposition 17.2, every fixed plane of such an element is a conjugate of F . By Lemma 17.1, each such plane contains exactly 10 edges of P_{600} . Each edge spans a 2-plane through the origin, so each edge lies in exactly one such fixed plane. Thus 720 edges determine 72 such planes, and each such plane determines four nontrivial order 5 rotations. Therefore there are $72 \cdot 4 = 288$ such elements. $\square$

17.2. The lift group $\tilde{G}_{600}$ and the associated $\mathbb{Z}/2$ -extension. By Proposition 11.10, each vertex of Γ_{600} has valency 12, so the assignment $\mu_{600}([m_e]) = -1$ on edge meridians is well-defined. Let $p_{600} : \widetilde{M}_{600} \rightarrow M_{600}$ be the associated connected double cover corresponding to $H_{600} := \ker(\mu_{600})$, let τ_{600} denote its nontrivial deck involution, and let $\mathcal{I}_{600} \rightarrow M_{600}$ be the associated real line bundle. Define

$$\tilde{G}_{600} := \left\{ \tilde{g} \in \operatorname{Homeo}(\widetilde{M}_{600}) \mid \exists g \in G_{600} \text{ such that } p_{600} \circ \tilde{g} = g \circ p_{600} \right\}.$$

PROPOSITION 17.4 (Splitting for the 600-cell group). *The G_{600} -action on M_{600} lifts to a fiberwise linear action by bundle automorphisms: for each*

$g \in G_{600}$, there is a bundle automorphism $\widehat{g} : \mathcal{I}_{600} \rightarrow \mathcal{I}_{600}$ covering g , and $\widehat{gh} = \widehat{g} \circ \widehat{h}$ for all $g, h \in G_{600}$.

PROOF. Since every $g \in G_{600}$ preserves Γ_{600} and permutes its edges, Lemma 12.2 gives $\mu_{600} \circ g_* = \mu_{600}$. Hence, by Lemma 12.3, every $g \in G_{600}$ admits a lift to $\widetilde{M}_{600}$, unique up to composition with τ_{600} . By Lemma 12.4, one therefore has a short exact sequence

$$1 \longrightarrow \langle \tau_{600} \rangle \longrightarrow \widetilde{G}_{600} \xrightarrow{\pi} G_{600} \longrightarrow 1.$$

It suffices to construct a splitting $s : G_{600} \rightarrow \widetilde{G}_{600}$. Choose generators $\sigma_1, \sigma_2, \sigma_3 \in G_{600}$ as in Equation (11.5), with presentation

$$G_{600} \cong [3, 3, 5]^+ \\ = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^3 = \sigma_2^3 = \sigma_3^5 = (\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2 = 1 \rangle.$$

Step 1: choose lifts of $\sigma_1, \sigma_2, \sigma_3$ with the same odd orders upstairs. By Lemma 12.3, each σ_i admits a lift to $\widetilde{M}_{600}$. Choose arbitrary lifts $\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3$. Since $\sigma_1^3 = \sigma_2^3 = \sigma_3^5 = 1$, the elements $\tilde{\sigma}_1^3, \tilde{\sigma}_2^3, \tilde{\sigma}_3^5$ are deck transformations, hence lie in $\{\text{id}, \tau_{600}\}$. If $\tilde{\sigma}_1^3 = \tau_{600}$, replace $\tilde{\sigma}_1$ by $\tau_{600}\tilde{\sigma}_1$. Since every lift commutes with τ_{600} , $(\tau_{600}\tilde{\sigma}_1)^3 = \text{id}$. The same odd-order adjustment applies to $\tilde{\sigma}_2$ and $\tilde{\sigma}_3$. Thus we may assume $\tilde{\sigma}_1^3 = \tilde{\sigma}_2^3 = \tilde{\sigma}_3^5 = \text{id}$.

Step 2: verify the involution relations upstairs. Set $\tilde{\sigma}_1\tilde{\sigma}_2, \tilde{\sigma}_2\tilde{\sigma}_3$, and $\tilde{\sigma}_1\tilde{\sigma}_2\tilde{\sigma}_3$ as lifts of $\sigma_1\sigma_2, \sigma_2\sigma_3$, and $\sigma_1\sigma_2\sigma_3$. Since $(\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2 = 1$ in G_{600} , each of $(\tilde{\sigma}_1\tilde{\sigma}_2)^2, (\tilde{\sigma}_2\tilde{\sigma}_3)^2$, and $(\tilde{\sigma}_1\tilde{\sigma}_2\tilde{\sigma}_3)^2$ is a deck transformation. We claim that each of $\sigma_1\sigma_2, \sigma_2\sigma_3$, and $\sigma_1\sigma_2\sigma_3$ has a fixed point in M_{600} . First, none of them is equal to $-\text{id}$. Indeed, if $\sigma_1\sigma_2 = -\text{id}$, then $\sigma_1 = -\sigma_2^{-1}$, so $\sigma_1^3 = (-\sigma_2^{-1})^3 = -\text{id}$, contrary to $\sigma_1^3 = \text{id}$. If $\sigma_2\sigma_3 = -\text{id}$, then $\sigma_2 = -\sigma_3^{-1}$, so $\sigma_2^{15} = (-\sigma_3^{-1})^{15} = -\text{id}$, contrary to $\sigma_2^{15} = \text{id}$. If $\sigma_1\sigma_2\sigma_3 = -\text{id}$, then $\sigma_1\sigma_2 = -\sigma_3^{-1}$, but $\sigma_1\sigma_2$ has order dividing 2, whereas $-\sigma_3^{-1}$ has order 10, a contradiction.

Now let $g \in \{\sigma_1\sigma_2, \sigma_2\sigma_3, \sigma_1\sigma_2\sigma_3\}$. If $g = \text{id}$, then trivially $\text{Fix}(g) \cap M_{600} = M_{600} \neq \emptyset$. Assume $g \neq \text{id}$. Then g is an involution in $SO(4)$ distinct from $\pm \text{id}$. Hence its $+1$ - and -1 -eigenspaces are both 2-dimensional, and therefore $\text{Fix}(g|_{S^3})$ is a great circle. We claim that $\text{Fix}(g|_{S^3}) \not\subset \Gamma_{600}$. If $\text{Fix}(g|_{S^3}) \subset \Gamma_{600}$, then this great circle contains an open arc of the radial image of some edge $E = [u, v]$ of P_{600} . The map g fixes this open arc pointwise, and hence fixes $W_E = \text{span}_{\mathbb{R}}\{u, v\}$ pointwise. Therefore g fixes E pointwise. But this is impossible by Proposition 17.2, since the orientation-preserving pointwise stabilizer of any edge is cyclic of order 5. Therefore, $\text{Fix}(g|_{S^3}) \cap M_{600} \neq \emptyset$.

Choose $x_{12} \in \text{Fix}(\sigma_1\sigma_2) \cap M_{600}$, $x_{23} \in \text{Fix}(\sigma_2\sigma_3) \cap M_{600}$, and $x_{123} \in \text{Fix}(\sigma_1\sigma_2\sigma_3) \cap M_{600}$, and lifts $\tilde{x}_{12} \in p_{600}^{-1}(x_{12}), \tilde{x}_{23} \in p_{600}^{-1}(x_{23})$, and $\tilde{x}_{123} \in p_{600}^{-1}(x_{123})$. Since $\tilde{\sigma}_1\tilde{\sigma}_2$ covers $\sigma_1\sigma_2$ and $\sigma_1\sigma_2(x_{12}) = x_{12}$, the point $\tilde{\sigma}_1\tilde{\sigma}_2(\tilde{x}_{12})$ lies in the two-point fiber $p_{600}^{-1}(x_{12}) = \{\tilde{x}_{12}, \tau_{600}(\tilde{x}_{12})\}$. Hence $(\tilde{\sigma}_1\tilde{\sigma}_2)^2(\tilde{x}_{12}) = \tilde{x}_{12}$. Since $(\tilde{\sigma}_1\tilde{\sigma}_2)^2$ is a deck transformation and the nontrivial deck involution τ_{600} acts freely, it follows that $(\tilde{\sigma}_1\tilde{\sigma}_2)^2 = \text{id}$. Similarly, $(\tilde{\sigma}_2\tilde{\sigma}_3)^2(\tilde{x}_{23}) = \tilde{x}_{23}$

and $(\tilde{\sigma}_1\tilde{\sigma}_2\tilde{\sigma}_3)^2(\tilde{x}_{123}) = \tilde{x}_{123}$. Since both squares are deck transformations, the same argument gives $(\tilde{\sigma}_2\tilde{\sigma}_3)^2 = (\tilde{\sigma}_1\tilde{\sigma}_2\tilde{\sigma}_3)^2 = \text{id}$.

Thus, the chosen lifts satisfy all defining relations of Equation (11.5). Therefore the map

$$\sigma_1 \mapsto \tilde{\sigma}_1, \quad \sigma_2 \mapsto \tilde{\sigma}_2, \quad \sigma_3 \mapsto \tilde{\sigma}_3$$

extends to a homomorphism $s : G_{600} \rightarrow \tilde{G}_{600}$ with $\pi \circ s = \text{id}_{G_{600}}$. Hence, by Lemma 12.4, the extension splits. For each $g \in G_{600}$, let $\hat{g} : \mathcal{I}_{600} \rightarrow \mathcal{I}_{600}$ be the fiberwise linear bundle automorphism induced by the lift $s(g)$ via Lemma 12.5. Since s is a homomorphism, these bundle maps satisfy $\widehat{gh} = \hat{g} \circ \hat{h}$ for all $g, h \in G_{600}$. This proves the proposition. $\square$

References

- [1] Gorapada Bera and Thomas Walpuski. Dirac operators twisted by ramified euclidean line bundles. *arXiv preprint arXiv:2503.01392*, 2025.
- [2] Jiahuang Chen. Perturbation and pruning of nondegenerate $\mathbb{Z}/2$ harmonic 1-forms. *arXiv preprint arXiv:2508.16187*, 2025.
- [3] Jiahuang Chen and Siqi He. On the existence and rigidity of critical $\mathbb{Z}/2$ eigenvalues. *arXiv preprint arXiv:2404.05387*, 2024.
- [4] H. S. M. Coxeter. *Introduction to Geometry*. John Wiley & Sons, New York, 2nd edition, 1969. MR 0346644
- [5] Harold Scott Macdonald Coxeter. *Regular polytopes*. Courier Corporation, 1973. MR 0370327
- [6] Michael W. Davis. *The Geometry and Topology of Coxeter Groups*, volume 32 of *London Mathematical Society Monographs Series*. Princeton University Press, 2008. MR 2360474
- [7] Harold Donnelly and Charles Fefferman. Nodal sets of eigenfunctions on Riemannian manifolds. *Inventiones Mathematicae*, 93(1):161–183, 1988. MR 0943927
- [8] Federico Franceschini, Rafe Mazzeo, and Paul Minter. Minimal surfaces with stratified branching sets. *arXiv preprint arXiv:2603.27168*, 2026.
- [9] Federico Franceschini and Ovidiu Savin. Solutions to the thin obstacle problem with non-2D frequency, 2025. arXiv:2504.16486 [math.AP].
- [10] David Gilbarg and Neil S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag, Berlin, 2nd ed., revised 3rd printing edition, 1998. MR 0473443
- [11] Qing Han. Singular sets of solutions to elliptic equations. *Indiana University Mathematics Journal*, 43(3):983–1002, 1994. MR 1305956
- [12] Qing Han, Robert Hardt, and Fanghua Lin. Geometric measure of singular sets of elliptic equations. *Communications on Pure and Applied Mathematics*, 51(11–12):1425–1443, 1998. MR 1639155
- [13] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, Cambridge, 2002. MR 1867354
- [14] Andriy Haydys, Rafe Mazzeo, and Ryosuke Takahashi. An index theorem for $\mathbb{Z}/2$ -harmonic spinors branching along a graph, 2023. arXiv:2310.15295 [math.DG].
- [15] Andriy Haydys, Rafe Mazzeo, and Ryosuke Takahashi. New examples of $\mathbb{Z}/2$ harmonic 1-forms and their deformations, 2023. arXiv:2307.06227 [math.DG]. MR 4873875
- [16] Dale Husemöller. *Fibre bundles*, volume 5. Springer, 1966. MR 0229247
- [17] Norman W Johnson and Asia Ivić Weiss. Quadratic integers and coxeter groups. *Canadian Journal of Mathematics*, 51(6):1307–1336, 1999. MR 1756885
- [18] Dmitry Kozlov. *Combinatorial algebraic topology*. Springer, 2008. MR 2361455

- [19] Mihai Nica. Eigenvalues and eigenfunctions of the laplacian. *The Waterloo Mathematics Review*, 1(2):23–34, 2011.
- [20] John Stillwell. *Naive lie theory*. Springer Science & Business Media, 2008. MR 2431402
- [21] Ryosuke Takahashi. Index theorem for $\mathbb{Z}/2$ -harmonic spinors. *Mathematical Research Letters*, 25(5):1645–1671, 2018. MR 3917743
- [22] Clifford Henry Taubes. Compactness theorems for $SL(2;\mathbb{C})$ generalizations of the 4-dimensional anti-self dual equations, 2013. arXiv:1307.6447 [math.DG].
- [23] Clifford Henry Taubes. The zero loci of $\mathbb{Z}/2$ harmonic spinors in dimension 2, 3 and 4, 2014. arXiv:1407.6206 [math.DG].
- [24] Clifford Henry Taubes. On the behavior of sequences of solutions to $U(1)$ Seiberg–Witten systems in dimension 4, 2016. arXiv:1610.07163 [math.DG].
- [25] Clifford Henry Taubes. The behavior of sequences of solutions to the Vafa–Witten equations, 2017. arXiv:1702.04610 [math.DG].
- [26] Clifford Henry Taubes and Yingying Wu. Examples of singularity models for $\mathbb{Z}/2$ harmonic 1-forms and spinors in dimension 3. *Gökova Geometry-Topology Journal*, 2020. MR 4251085
- [27] Clifford Henry Taubes and Yingying Wu. Topological aspects of $\mathbb{Z}/2$ eigenfunctions for the Laplacian on S^2 . *Journal of Differential Geometry*, 128(1):379–462, 2024. MR 4773187

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MA
Email address: `chtaubes@math.harvard.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON, HOUSTON, TX
Email address: `ywu68@uh.edu`